ECONOMICS
403

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Lecture 1

Static Analysis of Aggregate Demand

1. The purpose of this lecture is to develop the rudiments of demand analysis. The techniques that are used in what follows can readily be applied to any equilibrium-type model to yield the relevant version of the familiar multipliers. Although the emphasis in what follows is analytic, some care is taken to develop the natural economic interpretation of what appear to be complicated algebraic formulas.

2. We start with the basic Keynesian-type expenditure model to illustrate the techniques and interpretation of the multipliers associated with a macro model. Consider:

$$Y = C(Y) + I + G, \tag{1}$$

where Y is the basic good of the economy, C is consumption by the private sector, I is investment by the private sector and G is government spending. It is always imperative to determine at the outset which are the exogenous or pre-determined variables and which are the endogenous variables, that is the variables determined by the model under analysis. Here the exogenous variables are I and G. Whether a variable is exogenous or endogenous is a feature of the model: I, for example, in the next model to be analysed is endogenous. In general, a model is said to be identified if the endogenous variables can be solved in principle as a function of the exogenous variables. For our purposes, we generally assume that the model is properly identified when the number of equations corresponds to the number of endogenous variables. Accordingly, we treat the expenditure model as identified with (1) serving to determine the level of output Y. Actually, we should keep in mind that this model is really short-hand for a traditional supply (AS) and demand (AD) model:

$$AD = C(AS) + I + G, \tag{2}$$

$$AS = AD. \tag{3}$$

We let Y stand for AS, and assume implicitly that supply meets demand when (1) is considered. Now how to determine Y? One approach is to assume that consumption is linear in Y:

$$C(Y) = c_0 + c_1 Y, \tag{4}$$

where C_1 is the marginal propensity to consume out of income (MPC). For this formulation of the consumption function we have

$$c_1 = \frac{\Delta C}{\Delta Y} = \frac{\partial C}{\partial Y}; \tag{5}$$

that is, an additional unit of income generates c_1 units of consumption. It should be kept in mind that MPC is constant for consumption that is linear in income. The model now can readily be solved: by substitution,

$$Y = (c_0 + c_1 Y) + I + G \Rightarrow Y = \frac{(c_0 + I + G)}{(1 - c_1)}.$$

Notice that the multipliers for this model can readily be read off the explicit equilibrium solution:

$$\frac{\partial Y}{\partial G} = \frac{\partial Y}{\partial I} = \frac{1}{1 - c_1}. \tag{6}$$

An additional unit of exogenous investment causes equilibrium output to rise by the value of the multiplier (e.g., when $c_1 = .9$, Y increases by 10 in response to a unit increase in G).

The form of the expression (6) and the role of c_1 in this expression can be motivated by the following suggestive argument. Consider an increase in investment given by dI:

First round change in output:	dI	direct investment
Second round change in output:	$c_1 dI$	multiplier effect of first round
Third round change in output:	$c_1(c_1 dI) = c_1^2 dI$	multiplier effect of second round
Nth round change in output:	$c_1^n dI$	multiplier effect of previous round

Now sum up to infinity to obtain:

$$dY = dI(1 + c_1 + c_1^2 + ... + c_1^n + ...) = \frac{dI}{1 - c_1}.$$

A third approach to determining the multiplier is to take total differentials of the equilibrium condition (2): where $C(Y)$ is linear in Y, we have

$$dY = dc_0 + c_1 dY + dc_1 Y + dI + dG.$$

Llet $dc_0 = dc_1 = dG = 0$. It follows that $\frac{\partial Y}{\partial I} = \frac{1}{1-c_1}$ as before. The importance of this argument is that it can be used even if the function $c(Y)$ is not linear in Y. Suppose we write the equilibrium condition more generally as $Y = c(Y) + I + G$. Again take total differentials to find

$$dY = c_Y dY + dI + dG, \tag{7}$$

with $c_Y = \frac{\partial C}{\partial Y}$.We have then $\frac{\partial Y}{\partial I} = \frac{1}{1-c_Y}$, which generalizes the previous expression for the multiplier to the nonlinear case. It should be noted that in this general context C_Y is itself a function of Y and in more careful notation would be written as $C_Y(Y)$.

3. We now turn to an analysis of the IS curve which corresponds to a simple enrichment of the previous expenditure model:

$$Y = c(Y) + I(r) + G. \tag{8}$$

Here the endogenous variable remains Y, but the exogenous variables are now G and r, the interest rate. We assume that $1 > c_Y > 0$ that is, an increase in income leads to greater consumption no matter the level of income, and the increase in consumption is not greater than the increase in income and $I_r < 0$ that is, an increase in the interest rate leads to lower investment by the private sector .Now take total differentials of (8) to obtain:

$$dY = c_Y dY + I_r dr + dG. \tag{9}$$

First suppose $dG = 0$. We have $dr = (\frac{1-c_Y}{I_r})dY$. The expression in square brackets is the slope of the usual IS curve in (r,Y) space which is negative due to the assumptions concerning the signs of c_Y and I_r; see Figure 1a. We have the usual argument: if r declines, then investment goes up, so income is up and consumption increases with further increases in output. A more formal argument runs as follows. Rewrite the previous expression as:

$$dY = \frac{1}{1-c_Y} I_r dr. \tag{10}$$

If r changes by dr, investment changes by $I_r dr$; that is, a change in the level of interest leads to a change in I and hence Y by this amount. Now

the consumption multiplier from the previous model kicks in, and we are back to expression (10).

Government spending is a shift parameter for this curve drawn in (r,Y) space. Suppose r is fixed and G increases by dG ; from (9) we have $dY = \left[\frac{1}{1-c_Y}\right] dG$ which is the multiplier from the basic expenditure model $r$is fixed, so I is fixed see Figure 1b for a graphical summary, where dY = *(Y1 - Y0).*

4. We now add money to the economy. Let the supply of money be denoted by M. Let the demand for real money that is, $\frac{M}{P}$ where P is the price level be a function of income Y and the interest rate r. Since prices do not as yet play a role in this model we can set the price level equal to 1. So the money market equilibrium condition whereby the demand for money equals the supply of money is captured by the equation:

$$m(Y,r) = M, \tag{11}$$

where $m_Y > 0$ and $m_r < 0$. Since r is the opportunity cost of holding money, m_r is taken to be negative; the sign of m_Y is somewhat more problematic, but is taken to be positive to reflect a transaction need. Take the total differential of (11) to find

$$dM = m_Y dY + m_r dr. \tag{12}$$

Suppose M is fixed. It follows that $dr = \left(-\frac{m_y}{m_r}\right) dY$. The expression in square brackets is the slope of the traditional LM curve; see Figure 2a. The level of money is a shift parameter for the LM curve. Fix r and let M increase by dM. From (11), we see that $dY = \left(\frac{1}{m_y}\right) dM$ with dY = *(Y - Y*')*; see Figure 2b.

5. Consider now the goods and money market together.

$$Y = c(Y) + I(r) + G, \tag{13}$$

$$M = m(r, Y). \tag{14}$$

Here the endogenous variables are taken to be Y and r, while the exogenous variables are G (in some sense, representing a fiscal policy variable) and M (representing a monetary policy variable). As usual, take total differentials,

collecting terms in dY, dr (the endogenous variables) on the left and terms in dG , dM on the right:

$$(1 - c_Y)dY - I_r dr = dG, \tag{15}$$

$$m_Y dY + m_r dr = dM; \tag{16}$$

or in matrix form:

$$\begin{pmatrix} 1 - c_Y & -I_r \\ m_Y & m_r \end{pmatrix} \begin{pmatrix} dY \\ dr \end{pmatrix} = \begin{pmatrix} dG \\ dM \end{pmatrix}. \tag{17}$$

Using (17), it is straightforward to determine the direction of certain policy initiatives. We consider two cases.

case 1 $dG \neq 0$; $dM = 0$ apurely fiscal initiative Divide both equations by dG to obtain:

$$\begin{pmatrix} 1 - c_Y & -I_r \\ m_Y & m_r \end{pmatrix} \begin{pmatrix} \frac{\partial Y}{\partial G} \\ \frac{\partial r}{\partial G} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{18}$$

Now we can use Cramer's rule to derive the multipliers.

$$\frac{\partial Y}{\partial G} = \frac{\begin{vmatrix} 1 & -I_r \\ 0 & m_r \end{vmatrix}}{\begin{vmatrix} 1 - c_Y & -I_r \\ m_Y & m_r \end{vmatrix}} = \frac{m_r}{m_r(1 - c_Y) + m_Y I_r} = \frac{1}{(1 - c_Y) + \frac{m_Y}{m_r} I_r} = \frac{1}{\Delta}, \tag{19}$$

$$\frac{\partial r}{\partial G} = \frac{\begin{vmatrix} 1 - c_Y & 1 \\ m_Y & 0 \end{vmatrix}}{\begin{vmatrix} 1 - c_Y & -I_r \\ m_Y & m_r \end{vmatrix}} = \frac{-m_Y}{m_r(1 - c_Y) + m_Y I_r} = \frac{-\frac{m_Y}{m_r}}{(1 - c_Y) + I_r \frac{m_Y}{m_r}} = \frac{-\frac{m_Y}{m_r}}{\Delta}. \tag{20}$$

Notice that by the assumptions on the signs of the partial derivatives of the consumption and money demand functions , Δ^{-1} is positive as is $\frac{\partial Y}{\partial G}$, but $\frac{\partial r}{\partial G}$ is positive. The graphical story is given by Figure 3. When G rises, output would go to Y_1 if interest rates were fixed. But there is disequilibrium in the money market; in particular, with increased output there is excess demand for money. Interest rates must rise in response. Consequently, the rise is dampened by lower investment and equilibrium is reached at Y_2.

Can any sense be made of the relatively complicated expression Δ^{-1}? A logical analysis is revealing. With an increase in government spending dG, there is a round 1 increase in output given by $dY = dG$. There are two second round consequences. First, as in the expenditure model, consumption increases by $c_Y dG$. But interest rates must rise as a result of the Round 1 increase in output. By how much? From the LM curve, we know that $dr = -\frac{m_Y}{m_r} dY$; and the change in r impacts on output through a change in investment $dI = I_r dr = -\frac{I_r m_Y}{m_r} dG$, where it should be repeated that dG represents the Round 1 change in output. It follows that the Round 2 change in output is given by the sum of the consumption effect and the investment effect: $(c_Y - \frac{I_r m_Y}{m_r})dG$. As in the analysis in section 2 above indicated, the Round 3 effect is given by $(c_Y - \frac{I_r m_Y}{m_r})^2 dG$. Summing the effects over all the rounds to infinity gives $dY = \Delta^{-1} dG$ as derived above. With respect to the impact on interest rates, notice that from Figure 3, $dr =$ slope of the LM curve $dY = = (-\frac{m_Y}{m_r})\Delta^{-1} dG$.

case 2 $dG = 0; dM \neq 0$.

We follow the same line of reasoning as in Case 1. Divide (17) by dM to obtain:

$$\begin{pmatrix} 1 - c_Y & -I_r \\ m_Y & m_r \end{pmatrix} \begin{pmatrix} \frac{\partial Y}{\partial M} \\ \frac{\partial r}{\partial M} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{21}$$

Again use Cramer's rule to find the multipliers.

$$\frac{\partial Y}{\partial M} = \frac{\begin{vmatrix} 0 & -I_r \\ 1 & m_r \end{vmatrix}}{\begin{vmatrix} 1 - c_Y & -I_r \\ m_Y & m_r \end{vmatrix}} = \frac{I_r}{m_r(1 - c_Y) + m_Y I_r} = \frac{\frac{I_r}{m_r}}{(1 - c_Y) + \frac{m_Y}{m_r} I_r} = \frac{\frac{I_r}{m_r}}{\Delta}, \tag{22}$$

$$\frac{\partial r}{\partial M} = \frac{\begin{vmatrix} 1 - c_Y & 0 \\ m_Y & 1 \end{vmatrix}}{\begin{vmatrix} 1 - c_Y & -I_r \\ m_Y & m_r \end{vmatrix}} = \frac{1 - c_Y}{m_r(1 - c_Y) + m_Y I_r} = \frac{\frac{1 - c_Y}{m_r}}{(1 - c_Y) + \frac{m_Y}{m_r} I_r} = \frac{\frac{1 - c_Y}{m_r}}{\Delta}. \tag{23}$$

Notice first that an increase in the money supply will increase Y and decrease r; with an increase in M, there is an excess supply of money and equilibrium in the money market is only re- established with a fall in r; this fall leads to an increase in investment and triggers the multiplier effect analysed in the previous case. Figure 4 summarizes the argument.

The complex expression for $\frac{\partial Y}{\partial M}$ can readily be motivated. An increase in M depresses r by an amount given by equation (12): $dr = \frac{1}{m_r} dM$. The change in r affects investment by $dI = I_r dr = \frac{I_r}{m_r} dM$. This corresponds to a Round 1 effect on output. We now repeat the argument of the previous case to establish that the final impact of a change in M on output is given by $dY = (\frac{I_r}{m_r})\Delta^{-1} dM$, which confirms the above analytic argument. Finally, notice that the change dr is related to dY by the slope of the IS curve: $dr =$ slpe IS curve $dY = (\frac{1-c_Y}{I_r})(\frac{I_r}{m_r})\Delta^{-1} dM = (\frac{1-c_Y}{m_r})\Delta^{-1} dM$.

6. The Aggregate Demand schedule AD can be readily derived from the preceding analysis. First, the role of prices in the LM curve must be explicitly acknowledged. Equation (14) is rewritten as: $\frac{M}{P} = m(r, Y)$, so that the IS-LM system in differential form is given by:

$$\begin{pmatrix} 1 - c_Y & -I_r \\ m_Y & m_r \end{pmatrix} \begin{pmatrix} dY \\ dr \end{pmatrix} = \begin{pmatrix} dG \\ dM - M dP \end{pmatrix}, \tag{24}$$

where it should be kept in mind that the initial price level is 1. By the usual argument we find the multiplier associated with an exogenous change in prices (assuming $dG = dM = 0$):

$$\frac{\partial Y}{\partial P} = \frac{\begin{vmatrix} 0 & -I_r \\ -M & m_r \end{vmatrix}}{\begin{vmatrix} 1 - c_Y & -I_r \\ m_Y & m_r \end{vmatrix}} = \frac{-M I_r}{m_r(1 - c_Y) + m_Y I_r} = \frac{\frac{-M I_r}{m_r}}{(1 - c_Y) + \frac{m_Y}{m_r} I_r} = \frac{\frac{-M I_r}{m_r}}{\Delta}, \tag{25}$$

which is negative. Even though AD has a familiar shape, the negative slope doesn't follow from the usual substitution argument that gives negatively sloped demand curves in microeconomics. Here, as P changes, we have disequilibrium in the money market which, as usual, causes changes in the interest rate level which in turn affects the level of investment. This transmission mechanism is precisely what is described by the numerator of (25). For a graphical summary, see Figure 5.

7. There remain two important general technical points about macroeconomic models. The first concerns the issue of the existence and uniqueness of the equilibrium. Generally, as above, we are interested in determining the value of n endogenous variables $y_1, \ldots, y_n$ from a system comprising n

equations given by $f_1, ..., f_n$; as usual, there are some exogenous variables, say, $x_1, ..., x_m$. The system can be compactly written as $f_i(\ y_1, \ \ldots \ , y_n; x_1, ..., x_m) = 0$, $i = 1, ..., n$. The key relevant result in analysis is the implicit function theorem which states that unique solutions $y_i = y_i(x_1, ..., x_m)$ to the above system can be found as a function of the exogenous variables as long as the Jacobian of the system doesn't vanish. For a precise statement and some discussion of the theorem see Chiang pp. 204-214. As a reminder, the Jacobian of the system is defined to be the determinant of the $n \times n$ matrix $\left(\frac{\partial f_i}{\partial y_j}\right)$where $i, j = 1, ..., n$. For example, the Jacobian of the expenditure model system $Y = c(Y) + I + G$ is the 1×1 matrix $(1 - c_Y)$, which does not vanish as long as $c_Y \neq 1$. The Jacobian in the IS-LM system is the 2×2 matrix:

$$\begin{pmatrix} 1 - c_Y & -I_r \\ m_Y & m_r \end{pmatrix}.$$

Notice that in both models, we could not have derived the multipliers if the associated Jacobian had vanished. So at the heart of multiplier analysis given in this Lecture is the assumption that a unique solution to the system exists in the neighbourhood of the original equilibrium.

The second point concerns model stability. At point A in Figure 6 , there is excess supply in the goods market since interest rates are too high to support that level of output. Accordingly, suppliers will cut down on their holdings of inventories until equilibrium is reached at B. A similar story describes the move from C to D: at C, there is excess demand for goods and inventories run down forcing suppliers to increase output until equilibrium is reached at D. On the LM side, interest rates adjust if there is disequilibrium at E the excess supply of money drives interest rates down to F, while at G the scarcity of money leads to higher interest rates. The direction of these separate forces are combined in the IS-LM phase diagram which indicates that points away from the equilibrium level O have a tendency to move to O. Whether the movement is direct or revolves around equilibrium depends on the distance from and on the relative magnitude of the compensating forces originating in the goods and money markets. A more analytical argument can be found in Appendix 2.

8. The IS-LM model, which is a formal version of Keynes'General Theory advanced by the British nobel laureate Sir John Hicks of Keynes'General Theory, has had an important influence on economic policy in the capitalist

economies since WW II. There is an active role to be played by government represented in the theory by the variables G and M. In fact, a great deal of discussion in the sixties and seventies focused on the relative merits of these policy options in different circumstances. It must be said, however, that the model has severe limitations both methodologically and regarding specific economic assumptions. Regarding the latter, prices are assumed fixed, the government faces no budget constraint, the analysis ignores the production or supply side. Moreover, the analysis is purley static in nature and makes no attempt to chart the evolution of the econmy; agents are viewed as passive participants in the economy, incapable of any forward looking or strategic behaviour.

Appendix: A Short Review of Total Differentials

1. Students are advised to consult Chiang pp. 188-198 for a more complete presentation.

2. The total differential of an equation involving functions of one-variable involves no new ideas other than that of the total derivative. Suppose we are given the function $y = f(x)$. The total derivative of this function is simply given by:

$$\frac{dy}{dx} = f'(x) \text{ , or for small changes in } dx \text{ , } dy = f'(x)dx = f'dx \text{ .}$$

Now the usual rules of one-variable differentiation can be invoked to handle more complicated expressions.

$$y = f(x)g(x) \Rightarrow \frac{dy}{dx} = f'(x)g(x) + f(x)g'(x) \Rightarrow dy = f'gdx + fg'dx \text{ .}$$

The chain rule of differentiation often proves very useful.

$$y = f(g(x)) \Rightarrow \frac{dy}{dx} = f'(g(x))g'(x) \Rightarrow dy = f'g'dx \text{ .}$$

3. Consider the function of two variables given by $z = f(x, y)$. It takes a certain amount of work to establish the expression:

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \text{ .}$$

The overall change resulting from small changes in the basic variables dx, dy are given by a weighted sum of the these underlying changes where the weights are simply the partial derivatives in the appropriate directions. For example, suppose that $z = 3xy + 7\log(x)\,y^3$. Then

$$dz = \left[3y + \left(\frac{7}{x}\right)y^3\right]dx + \left[3x + 21\log(x)y^2\right]dy \text{ .}$$

4. For a macroeconomic application, consider the demand for real balances: $m(Y, r) = M/P$. It follows: $m_y dY + m_r dr = (dM)/P - (MdP)/P^2$. Generally, we assume that the initial equilibrium has $P = 1$, so that the equilibrium condition in the money market can be written:

$$m_y dY + m_r dr = dM - MdP.$$

Appendix: Stability in Macroeconomic Systems

1. In the models discussed in this Lecture the time path associated with an equilibrium value of output, to take an example, has been suppressed. All we know is the general direction of the adjustment process from an initial equilibrium given a small perturbation to the system. Ideally, there would be given some information concerning the magnitude of changes in this direction. Suppose then that a time-description of the evolution of the variable y is known as function of t i.e. $y = y(t) \equiv y_t$.

In rare circumstances, an explicit solution for y_t can be found. More likely, it is necessary to undertake a qualitative analysis involving what is called a phase diagram. The change $\frac{dy}{dt}$ is plotted against y alone with the other variables suppressed. It should be clear that at any point the locus crosses the horizontal axis, y^*, the system is in equilibrium: at y^* there is no movement in y, since $\frac{dy}{dt}$ is zero at that point. Now, the stability issue concerns what happens away from y^*. The answer again is straightforward: if the slope of the locus is negative at y^*, then the system is stable since to the left of y^* the variable y is positive and decreasing and to the right it is negative and increasing. Similarly, the system is unstable at the equilibrium y^*, if the slope of the locus of $\frac{dy}{dt}$ against y is positive at y^*. Consider two simple examples.

Example 1. Suppose $\frac{dy}{dt} = y^2 - 8y + 15 = (y-3)(y-5)$, which is graphed as Figure 7.

In this example, there are two equilibrium values, 3 and 5. Notice that the slope of the graph is negative at 3 but positive at 5; we conclude that the system is stable around 3 and unstable for y greater that 5. These qualitative observations are readily verified by computing $\frac{d}{dy}\left(\frac{dy}{dt}\right) = 2y - 8$; at 3, the slope is computed to be -2, while at 5, the slope is 2.

Example 2. Consider the differential equation $\frac{dy}{dt} = (3 - y) - \ln y$; the analysis of the qualitative properties of this equation is greatly simplified if the two components of the equation are graphed separately in Figure 8.

It is clear that to the left of y^* the locus $\frac{dy}{dt}$ is negative while to the right it is positive; accordingly, the system is stable around y^*.

2. In a system with two unknown variables say x and y, as in the IS-LM model presented in this lecture with Y and r, the phase diagram analysis is somewhat different. Here the system is analysed on the xy plane directly. Suppose the equilibrium is denoted by E. The directions of the movement of the x-variable is indicated in each of the four areas which are demarcated by the graphs of $\frac{dx}{dt} = 0$ i.e. $x' = 0$ and $\frac{dy}{dt}$ corresponding to the two equilibrium conditions dr conceptual direction, keep the IS-LM analysis in mind ;similarly the movement of the y-variable away from E is indicated. The combined directions are then analysed; some possibilities follow.

In Figure 9 all movement is away from E and the system is unstable. By contrast, in Figure 10 the movement is towards E in quadrants (1) and (3), while away from E in (2) and (4); such a pattern is called a saddle-path equilibrium and arises frequently in dynamic macroeconomic models. The following Figure 11 is an example of a stable focus (E here is a centre of movement), while E in the final figure is called a vortex.

Exercises

1. Consider the following version of the IS-LM model:

$$\begin{aligned} Y &= C(Y - T(Y)) + G + I(r) - IM(Y) + E\,, \\ \frac{M}{P} &= M(Y, r)\,; \end{aligned}$$

where IM is imports and E is exports, and $T(Y)$ are taxes.
a) What are the exogenous variables? the endogenous variables? What is the basic government expenditure multiplier in this model? Give an intuitive rationale for this expression.
b) Suppose there is a shift in the level of exports. What happens to the level of imports? to the interest rate? Show graphically and analytically.
c) Compute the slope of the AD (aggregate demand) curve associated with this model.
d) Show graphically the interdependence of the goods market, the money market and the labour market when there is a decrease in the level of exports.

2. Suppose taxes are fixed at T, where consumption is out of disposable income as in the previous problem. Suppose further that Ottawa wishes to cover an increase in spending with an offsetting increase in taxes. Find the change in output in the basic IS-LM framework and interpret the results.

3. Consider the following Keynesian-type model.

$Y = F(N)$	(production function)
$\frac{W^*}{P} = f(N)$	($f = F'$; nominal wages W * fixed)
$Y = c(Y) + I(r) + G$	(goods market equilibrium)
$\frac{M}{P} = M(r, Y)$	(money market equilibrium)

a) Compute ∂Y/∂G, ∂Y/∂M for this model. In what ways do the results differ from the multipliers associated with the AS-AD model considered in the lectures?
b) Give a simple definition of Keynesian unemployment for this model.

LECTURE I

Graphs

1. [Figures 1,2] The IS and LM curves are drawn in (r,Y) space. The negatively sloped IS curve shifts to the right when G increases. The LM curve is positively sloped, shifting to the right when M increases.

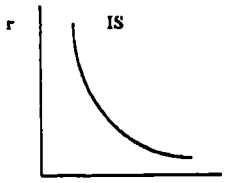


Figure 1a

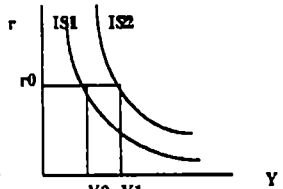


Figure 1b

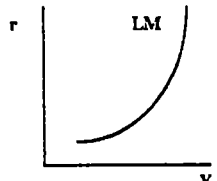


Figure 2a

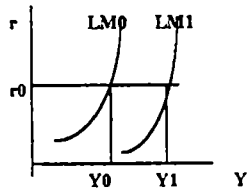


Figure 2b

2. [Figure 3] An increase in G shifts IS0 to IS1 with an increase in output. At (r0, Y1) there is excess demand for money and interest rates rise. This dampens investment and equilibrium output is now Y2. [Figure 4] The money supply increases shifting the LM curve outwards. At Y1 there is excess demand for money with the result that interest rates rise. Final output is at Y2.

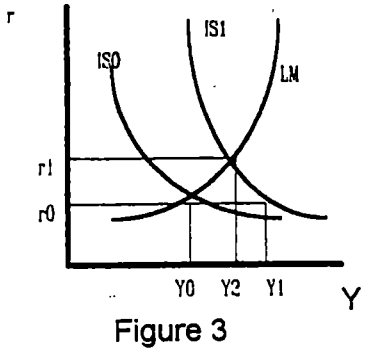


Figure 3

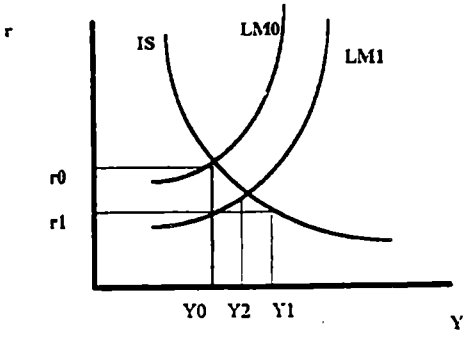


Figure 4

3. [Figure 5] The Aggregate Demand curve. A decrease in prices from P0 to P1 corresponds to an effective increase in the money supply with an increase in output resulting from greater levels of investment due to lower interest rates..

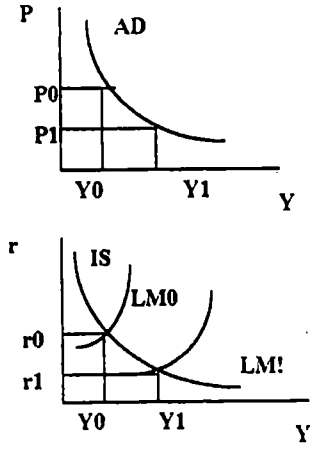
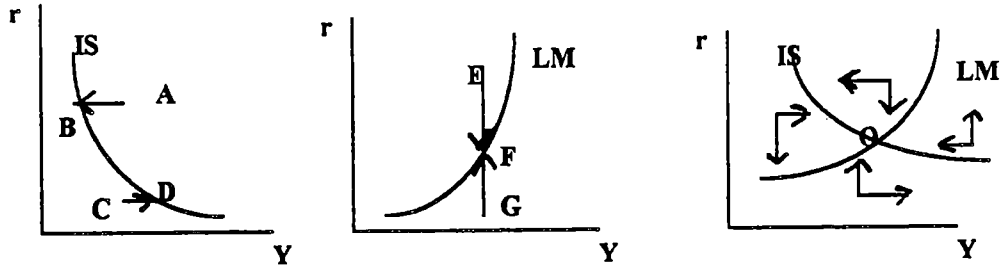
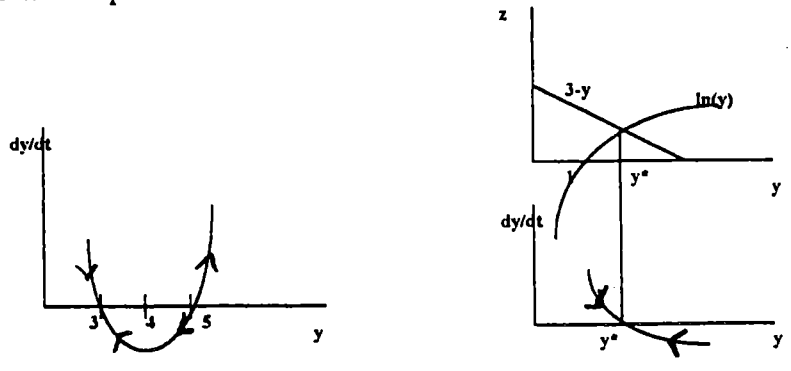


Figure 5

4. [Figure 6] The dynamics of the IS-LM analysis. At A there is excess supply of goods; inventories adjust back to B. At E there is excess supply of money and the price of bonds is bid up; ie interest rates drop. The economy converges to O.



5. [Figures 7, 8] In a one-variable phase diagram, the variable is dynamically stable if the slope is negative at the equilibrium.



6. [Figures 9-12] In Figure 9 all movement is away from E and the system is unstable. By contrast, in Figure 10 the movement is towrds E in quadrants (1) and (3), while away from E in (2) and (4); such a pattern is called a saddle-path equilibrium and arises frequently in dynamic macroeconomic models. The following Figure 11 is an example of a stable focus (E here is a centre of movement), while E in the final figure is called a vortex.

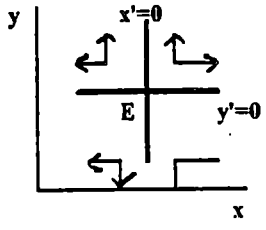


Figure 9

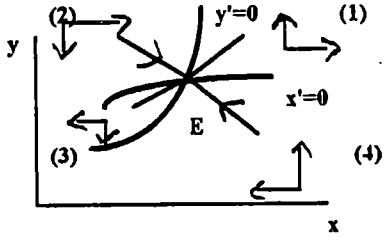


Figure 10

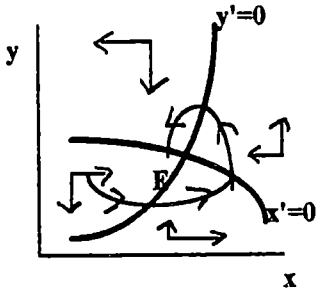


Figure 11

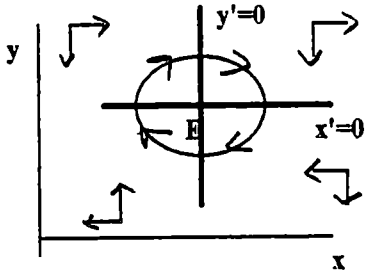


Figure 12

Lecture 2

Static Analysis of Aggregate Supply

1. In this Lecture, the production side of the economy is introduced to derive equilibrium conditions in the labour market which are the basis of the Aggrgate Supply curve (AS). An important theme is the role of expectations in determining a potential trade-off between price chages and output. The conceptual orientation of the material is derived from Milton Friedman's Nobel Lecture which was published in the American Economic Review in 1968 under the title *The Role of Monetary Policy.*

2. To analyze the demand for labour, we first introduce an economy-wide production function which ignores capital and treats output as a simple function of the amount of labour used; that is,

$$Y = F(N) \ , \tag{1}$$

where N is the number of labour inputs and $F' > 0$ but $F'' < 0$. Let $f \ (\equiv F')$ represent the marginal product of labour; by assumption $f' \ (\equiv F'') < 0$. The firm maximizes profits π by choosing optimal level of labour N:

$$\pi = P \cdot F(N) - W \cdot N \ , \tag{2}$$

where P is the price level and W is the nominal wage rate. As usual,

$$\frac{d\pi}{dN} = 0 \Rightarrow P \cdot F'(N) - W = 0 \Rightarrow f \ (N) = \frac{W}{P} \equiv w \quad ;$$

so that we have the standard microeconomic result that the firm hires labour up to the point where the marginal product of labour equals the exogenous wage rate (or the marginal physical product of labour equals the real wage rate). Notice that in what follows, lower-case letters are taken to represent

real values and upper-case nominal values. We write the demand for labour equation as:

$$W = P \cdot f(N)\,, \qquad f(N) < 0\,. \tag{3}$$

The graphic analysis can be given in either *(W, N)* space or *(w, N)* space; see Figure 1.

What happens if P_0 rises to P_1 ? For fixed W , the real wage rate goes down and so the firm can hire more labour; i.e. N goes up. Note carefully that this analysis takes different graphical form depending on whether we work in (w, N) space or *(W, N)* space.

3. The analysis of the supply of labour is somewhat trickier. To simplify matters we make two assumptions. First, we assume that more labour is supplied as the nominal wage rate increases to reflect the fact that leisure has become more expensive. Secondly, it is assumed that labour does not know the true price level P since firms set prices but have rather an expected value P^e in mind when offering labour. For the moment, we assume that P^e is exogenously given; the issue of how price expectations and expectations in general are to be analysed occupies a great deal of our attention later in the course. We simply posit a function of labour g with the property $g' > 0$ in the graphs that follow, P^e is represented by P^* ; see Figure 2 for a graphical representation..

4. Demand and supply of labour are now put together to give an analysis of aggregate supply (AS) in which we tentatively characterize P^e as function h of prices P. The most that can be asserted at this point about the properties of the expectation function h is that $0 \leq h' \leq 1$. Keep in mind that $dp^e = h'dp$: when h' is 0, price expectations are fixed even though prices may be moving; by contrast, if h' is 1,price expectations move in tandem with price changes. Our analysis of the labour market can be summarized then by the following equations:

$$W = P \cdot f(N)\,, \qquad f' < 0 \tag{4}$$

$$W = P^e \cdot g(N)\,, \qquad g' > 0 \tag{5}$$

$$P^e = h(P)\,; \qquad 0 \leq h' \leq 1 \tag{6}$$

which can be collapsed into the single equation which describes the aggregate supply side of the market (AS):

$$P \cdot f(N) = h(P) \cdot g(N) \tag{7}$$

Here there are two variables P, N which are both endogenous in the AD-AS analysis which ultimately follows. Output Y can always be retrieved from N via the production function. But first we should look at the shape of this formulation of the AS curve. Take total differentials of (7) to find:

$$dP \cdot f(N) + P \cdot f'(N)dN = h(P) \cdot g'(N)dN + h'(P) \cdot g(N)dP \ . \tag{8}$$

We assume that the system is initially in equilibrium and suppose that it has been so for a while, so that P^ecan be taken to be P which we further assume has been normalized to 1; it also follows from (7) in these circumstances that $g(N) = f(N) = w$. With these assumptions, the previous equation simplifies to:

$$(g' - f')dN = w(1 - h')dP \ , \tag{9}$$

which gives the slope of the AS curve:

$$\frac{dP}{dY} = \frac{dP}{dN} \cdot \frac{dN}{dY} = \frac{(g' - f')}{w(1 - h')} \cdot \frac{1}{f} \quad . \tag{10}$$

Notice that the slope is positive as long as $h' \neq 1$, in which the AS curve is vertical. It is instructive to analyse two extreme cases graphically: $h' = 1$, according to which P^e changes to match P; and $h' = 0$, where P^e is fixed. Figure 3 should be carefully analyzed.

The essential point of this analysis of the supply side of the economy, both analytic and graphical, is that the AS has positive slope only if the labour market does not have for some reason or other have the correct perception of upcoming price changes. In the argument of the right panel of Figure 3, labour was fooled and supplied more labour unwittingly. The argument should also be appreciated in the context where labour is not supplying enough labour because the price level is assumed to be higher than its true level. Suppose, for example, that the original equilibrium was at N3 in Figure 3. Prices fall, but labour does not react correctly to the fall in prices. As a result, labour supply falls to N0. It is important to realize that there is no involuntary unemployment in this situation. There is, what is termed, disguised unemployment represented by the difference between N1 and N0.

5. The demand and supply sides are merged to yield a complete analysis of the economy here characterized by the following four equations (with total differentials as indicated) in endogenous variables Y, P, r, N and exogenous variables G and M.

IS curve:

$$Y = c(Y) + I(r) + G\,,$$

$$\Rightarrow\ (1 - c_Y)dY - I_r dr = dG\,. \tag{11}$$

LM curve:

$$\frac{M}{P} = m(r, Y)\,,$$

$$\Rightarrow\ \frac{1}{P}dM - \frac{M}{P^2}dP = m_r dr + m_Y dY\,.$$

or, after setting initial price level to 1:

$$m_Y dY + m_r dr + M\,dP = dM\,. \tag{12}$$

Labour market:

$$P \cdot f(N) = h(P) \cdot g(N)\,,$$

$$\Rightarrow\ f(N)dP + P \cdot f'(N)dN = h'(P) \cdot g(N)dP + h(P)g'(N)dN\,,$$

$$\Rightarrow\ (1 - h') \cdot w dP + (f' - g')dN = 0\,; \tag{13}$$

where at the outset $h(P) = P$ and $f(N) = w$.

Production:

$$Y = F(N)\,,$$

$$\Rightarrow\ dY = f dN\,. \tag{14}$$

In the presence of the last equation, the system has three endogenous variables N, r, P and two exogenous variables G, M . The system can be organized in the following matrix form:

$$\begin{pmatrix} w \cdot (1 - c_Y) & -I_r & 0 \\ w \cdot m_Y & m_r & M \\ f' - g' & 0 & (1 - h') \cdot w \end{pmatrix} \begin{pmatrix} dN \\ dr \\ dP \end{pmatrix} = \begin{pmatrix} dG \\ dM \\ 0 \end{pmatrix} \tag{15}$$

which can be manipulated to yield the multipliers $\frac{\partial Y}{\partial G}$, $\frac{\partial r}{\partial G}$, $\frac{\partial Y}{\partial M}$, $\frac{\partial r}{\partial M}$.

For example,

$$\frac{\partial N}{\partial G} = \frac{\begin{vmatrix} 1 & -I_r & 0 \\ 0 & m_r & M \\ 0 & 0 & (1-h')\cdot w \end{vmatrix}}{\begin{vmatrix} w\cdot(1-c_y) & -I_r & 0 \\ w\cdot m_y & m_r & M \\ f'-g' & 0 & (1-h')\cdot w \end{vmatrix}}$$

$$= \frac{w\cdot m_r(1-h')}{w^2\cdot m_r(1-c_y)(1-h') - I_r(f'-g')M + w^2\cdot m_y I_r(1-h')}$$

$$= \frac{1}{w(1-c_y) + wI_r\frac{m_Y}{m_r} - \frac{M(f'-g')}{wm_r(1-h')}I_r} \qquad (16)$$

Since

$$\frac{\partial Y}{\partial G} = \frac{\partial Y}{\partial N}\cdot\frac{\partial N}{\partial G} = f\left(\frac{\partial N}{\partial G}\right) = w\left(\frac{\partial N}{\partial G}\right),$$

it follows that

$$\frac{\partial Y}{\partial G} = \frac{w}{w(1-c_Y) + wI_r\frac{m_Y}{m_r} - \frac{M(f'-g')}{wm_r(1-h')}I_r} = \frac{1}{(1-c_Y) + I_r\frac{m_Y}{m_r} - \frac{M(f'-g')}{w^2 m_r(1-h')}I_r} = \frac{1}{\Delta}. \qquad (17)$$

How to interpret this expression? First, it is instructive to look at the graphical account of what occurs when the level of government spending is increased in the context where AS has positive slope; see Figure 4.

The analysis has three components, the first two of which are already familiar. An increase in government spending shifts the IS curve to IS_1, an increase in output given by $dY = (Y_1 - Y_0) = (\frac{1}{1-c_Y})dG$. But at $Y_1$the money market is in disequilibrium and interest rates rise towards r_1 and output Y_2. Now

$$(Y_2 - Y_1) = \left(\frac{1}{1 - c_Y + I_r\frac{m_Y}{m_r}}\right)dG.$$

So far we are repeating the IS-LM analysis. However, at Y_2 there is an excess demand for goods, since the economy is unable to produce Y_2 at the price P_0 with the labour market in equilibrium. The price of goods is bid up with the real supply of money shrinking as a result until the price of output is such that the labour market is in equilibrium at P_1. Note that the LM

curve has shifted back to LM_1. Nominal wages have increased, but real wages must have gone down to have induced suppliers to have increased output. Finally, (Y_3 - Y_0) = $1/\Delta$, where Δ is given by equation (17). The first two terms in Δ are familiar in the way we have just argued; the clue to the last is also revealed in the graphical analysis and reflects the consequence of an increase in price due to disequilibrium in the labour market. It is interesting to pursue this point further.

6. We follow the type of logical analysis introduced in Sections 4 and 5 of the previous Lecture. The Round One change in output is simply dG. Round Two has three components. First, the change due increased consumption, the result of increased income: $c_Y dG$. Second, increased demand for money leads to an increase in interest rates $dr = (-\frac{m_Y}{m_r})dG$, and hence a change in investment spending: $-(I_r m_Y/m_r)dG$. The third effect originates in changing prices. First observe using (10):

$$\frac{dY}{dP} = \frac{dY}{dN}\frac{dN}{dP} = -\left(\frac{w^2(1-h')}{f'-g'}\right) , \tag{18}$$

since we are assuming that $f(n) = w$. It follows that the Round One increase in government spending has a consequence an increase in prices given by: $dP = -\left(\frac{(f'-g')}{w^2(1-h')}\right)dG$. But the stock of real balances is affected by the change in the price level:

$$d(M/P) = dM - MdP = -MdP = M\left(\frac{(f'-g')}{w^2(1-h')}\right)dG .$$

Again using the LM curve, effective changes in the stock of real money balances are related to interest rate movements to equilibrate the money market:

$$dr = \frac{1}{m_r}d(M/P) = \frac{1}{m_r}\left(\frac{M(f'-g')}{w^2(1-h')}\right)dG. \tag{19}$$

And we get back to output via investment, so that:

$$dY = \frac{I_r}{m_r}\left(\frac{M(f'-g')}{w^2(1-h')}\right)dG. \tag{20}$$

The cumulative Round Two effects then are the sum then of these three effects:

$$dY = \left[c_Y - I_r\frac{m_Y}{m_r} + \frac{I_r}{m_r}\left(\frac{M}{w^2}\frac{(f'-g')}{(1-h')}\right)\right]dG .$$

The process is iterated through infinity, and we find $\frac{\partial Y}{\partial G} = \frac{1}{\Delta}$.

6. What implications does a vertical AS curve have for IS-LM analysis? Recall that the endogenous variables for this model are r and Y. But a vertical supply curve fixes output at Y^* . Accordingly, the system is overdetermined with two equations specifying the value of the unknown interest rate. We must either drop an equation or introduce another endogenous variable. Figure 5 tells us how tp proceed. Equilibrium r^* can be read off the IS curve, with the balance of private sector and public sector spending so determined.. Now for the system to make any sense the LM curve must pass through (r^*, Y^*) . Since M is assumed fixed, it is necessary for prices to adjust to shift the curve from LM_0 to LM_1. So, in such a classical framework, the IS-LM framework degenerates into two independent equations, the one determing interest rates given the level of governmnet spending, the other determining the price level given the money supply. Put another way, the IS-LM contribuiton is to establish an interdependence between the goods market and the money market by fixing the price level and letting aggregate demand determine output.

7. It is clear from this analysis that as long as expected prices P^e match actual prices output will be Y* corresponding to a natural level of output for the underlying production function F. Aggregate Supply then is vertical at Y* as long as $P = P^e$ .When price P does not correspond to a given value of P^e, supply will be positively sloped. This in turn suggests a family of positively sloped AS curves indexed by the corresponding expectation level of prices P^e . A graphical summary is given in Figure 6; notice that AS_1, AS_2, AS_3 crosss the vertical AS curve when $P^e = P_1, P_2, P_3$ respectively.

In this context, suppose that there is a change in demand (due perhaps to changes in investment patterns) represented in Figure 6 by the shift in AD from AD_1 to AD_2 ; if price expectations remain fixed at P_1 , there is movement along the AS_1 curve to a new equilibrium at output level Y_2 and prices P_2. But this equilibrium can not be sustained as labour comes to realize that prices have changed and revises its projection of future prices. As price expectations change, so will the supply curves shifting from $AS_1$to AS_2 and so on to AS_3 where the actual price level corresponds to the expected price level at (of course) Y^* where the equilibrium price level is P_3.

At Y_2 there is disguised uneployment given by $Y^* - Y_2$ which is only reduced as price expectations are revised. Consider the following scheme for revising expectations which is called *adaptive expectations* in the economics literature:

$$P^e_{t+1} = P^e_t + \lambda(P_t - P^e_t) \tag{21}$$

The parameter λ in this equation is a measure of the speed of adjustment to past errors in making price expectations. If λ is zero, for example, then expecations are fixed at P^e_t, no matter the realized value of the price level; here expectations are said to be *static* . If λ is 1, then the current expectation is simply last period's value. The smaller the λ, the longer it will take expectations to adjust to the correct price level. In general, for $0 < \lambda < 1$, the expectation is a wieghted sum of all previous values of the price level with the weights appoaching zero in the infinite past; see Exercise 3.

In the context of expectation formation given by 21, it may be the case that it will take some time for agents to revise their expectations so that output returns to Y^* after the shock to demand given in Figure 6. Accordingly, the government could intervene by increasing its spending so that AD_3 shifts outward to pass through Y^* at a price level closer to P_2 . But there are drawbacks to this sort of intervention. The first is that it is tricky to execute, since the government must have a fairly precise reading of the impact of spending as well as the expectations formation process. The intervention does have real implications: government intervention effectively crowds out private investment. At (P_3, Y^*) the original mix of government spending and private investment has been re-established through lower interest rates caused by larger real balances as prices have fallen, whereas at (P_2, Y^*) there is less investment corresponding to the extra government spending. Another issue is whether the governmnet can maintain the level of public spending given its fiscal situation or whether it is compromising its long-run capacity to deal with more fundamental problems. Finally, it may be diificult for the government to resist the inflationary impulse with unfortunate consequences as now described.

Suppose, referring to Figure 7, that $\pi^e = 0$ and that the economy is on the appropriate AS curve. Assume further that a postive and one-time only demand shock has temporarily moved the economy to output Y_1.Unemployment is reduced as a consequence and positive effects are felt throughout the economy nd the government enjoys the political benefits. It certainly is tempting to maintain output at $Y_1$by inflating the economy; inflation increases from 0 to π_1 . But as agents adjust their price expecta-

tions so that $\pi^e = \pi_1$, the AS curve will shift upwards. Now to maintain output at Y_1 on AS($\pi^e = \pi_1$), which is now perhaps a policy objective, inflation must now increase to π_2 . Of course, the situation is not stable and the argument repeats itself with inflation now increasing to π_3 if the government is committed to maintaining output at Y_1 . All the while the AS curve shifts up. Eventually, the inflationary load is too difficult to bear and inflationary expectations will equal actual inflation; but at what level? According to the analysis, inflation will stabilize at π_3 . Now, the difficult business of actually reducing inflationary expectations in an orderly manner begins. If the government reduces inflation but the public remains sceptical and only slightly modifies its level of inflationary expectations, output will actually fall below Y^* . The road back to Y^* and $\pi = \pi^e = 0$ will be rocky enough.

8. The Aggregate Supply curve developed in this Lecture can be succinctly presented:

$$Y_t = \overline{Y} + \alpha(P - P^e) ,$$

where $\alpha > 0$. A worker-misperception story has been used to justify the model according to which exogenously fixed price perceptions may not correspond to actual prices with the consequence that the equilibrium real wage rate may be higher or lower than that at $\overline{Y}$. An important aspect of this account is that the labour market does clear : there is no involuntary unemployment. Workers simply have prices wrong and as soon as their price perceptions correspond to the actual price level output will return to the long-run level at $\overline{Y}$.

Contrast this story with one that emphasizes the role of labour contracts that prevail for several periods. The contract establishes a nominal wage for the duration of the contract according to a perceived price level. Labour may learn quite quicky that their price perceptions were inaccurate, but they will be unable to adjust their supply of labour in light of this information since they are locked in the contract. Here the AS curve will be described as above. The difference with the previous account is that the labour market does not clear during the later stages of the contract: nominal wages are fixed. A formal version of this story , *the sticky-wage model* , will be developed in Lecture 4 and a good deal of discussion in later Lectures will focus on non-equilibrium or non-Walrasian theories.

A production-side account of AS, formalized by Robert Lucas, dominated macroeconomic theorizing during the late 1970s.

Exercises

1. Suppose the AS curve has positive slope. Derive each of the multipliers $\frac{\partial Y}{\partial M}, \frac{\partial P}{\partial G}, \frac{\partial P}{\partial M}$ analytically. For each case, give a graphical analysis; for each case, give a logical analysis Round One effect, etc. to support the analytical result.

2. Derive all four multipliers for the case where the AS curve is vertical. Give the supportings graphical and logical analysis as in the previous problem.

3. Derive the two mutilpiers for the IS-LM model when output is fixed at Y^*.

4. Suppose that the expected value of a variable X is determined adaptively by the formula:

$$X_t^e = X_{t-1}^e + \lambda(X_{t-1} - X_{t-1}^e) .$$

Find values of w_i, $(i = 1, 2, ...)$, so that

$$X_t^e = \sum_{i=1}^{\infty} w_{t-i} X_{t-i} .$$

Lecture 2
Graphs

1. Figure 1: The Demand for Labour. The curve can be drawn in either (w,N) or (W,N) space. In both cases, the curve is negatively sloped since f (N) is negatively sloped; ie F"(N) < 0.

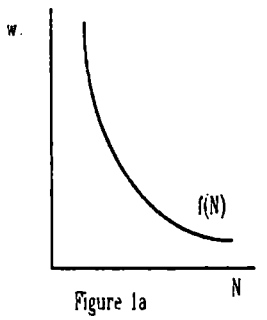


Figure 1a

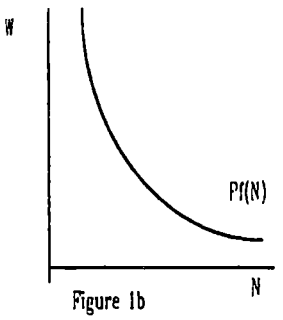


Figure 1b

Suppose nominal wages are fixed at W0 and prices fall from p0 tp p1. In Figure 1c, real wages fall from w0 to w1 and accordingly then demand for labour increases to N1. In (W,N) space, the analysis is somewht different. The increase in prices shifts the curve outwards; as in the previous analysis the new demand for labour is given by N1.

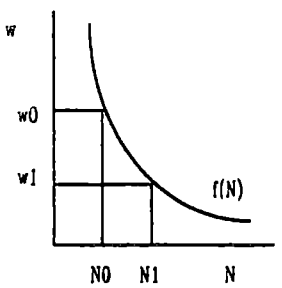


Figure 1c

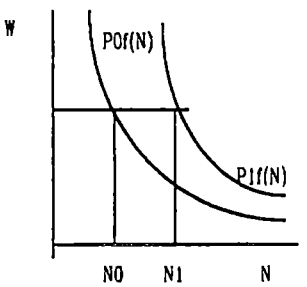


Figure 1d

2. Figure 2: The Supply of Labour. Workers offer supply depending on the anticipated or expected price level P*. For a fixed P*, the higher the nominal wage, the higher the expected real wage (ie W/P*) and the greater the supply of labour.

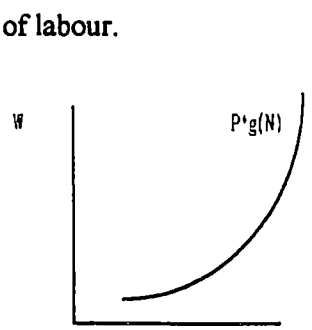


Figure 2a

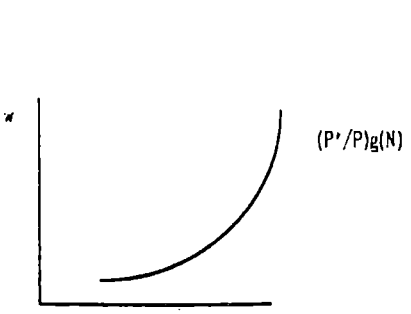
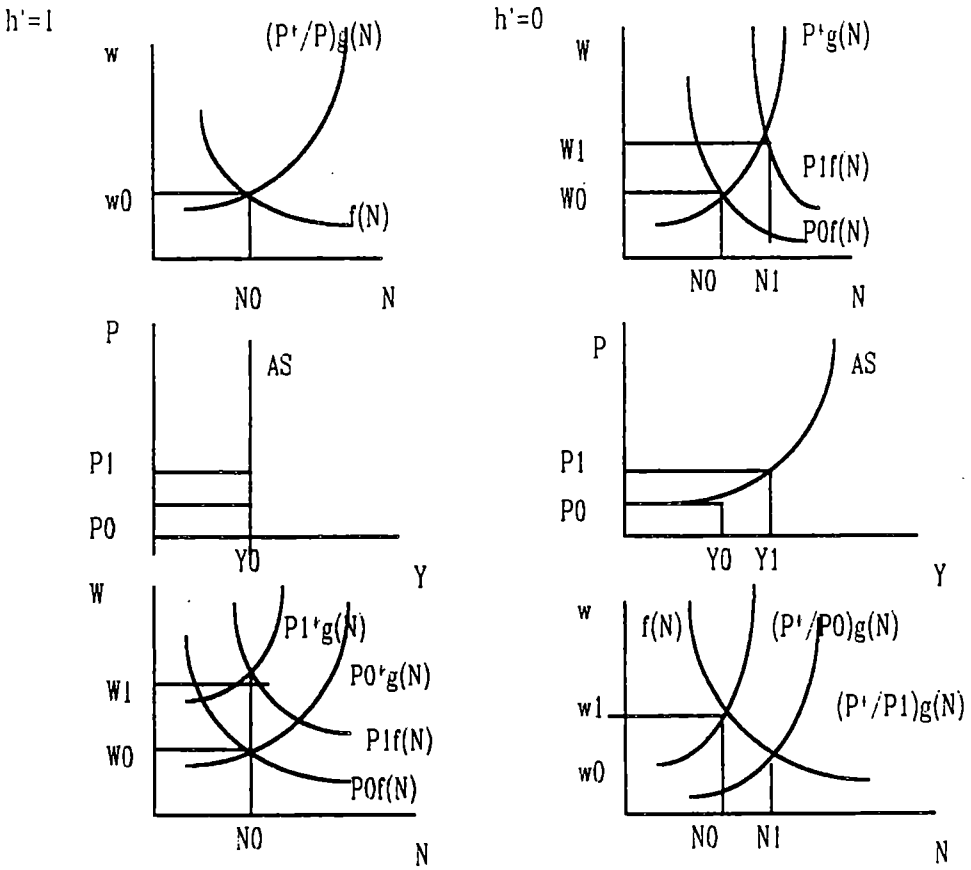


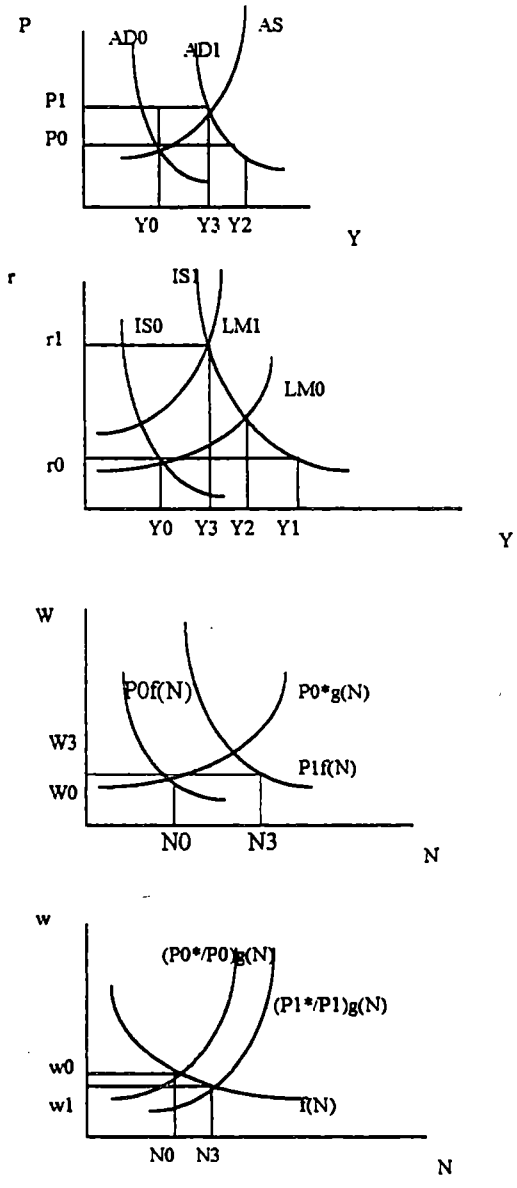
Figure 2b

3. Figure 3: The Aggregate Supply Curve. Suppose prices rise from p0 tp p1. The left panel assumes that price expectations perfectly match the change; ie h' = 1. In (W,N) space, both the supply and demand for labour shift outwards to offset the effect of the changes in prices and expectations; in (w,N) space, neither curve moves. The equlibrium demand for labour remains at N0 and the AS curve is vertical at Y0 = F(N0). The situation is different when price expectations do not move as far as the true price chnage; ie when h' = 0, they don't move at all. This extreme case is analyzed in the right panel. Here in (w,N) space, the supply of labour shifts down (since the ratio P*/P0 falls to P*/P1) and as a result more labour is offered for a given true real wage since labour falsely expects a larger real wage.

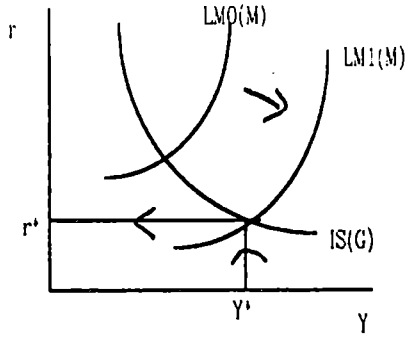


4. Figure 4: An Increase In Government Spending Under Static Expectations [ie, h' = 0]

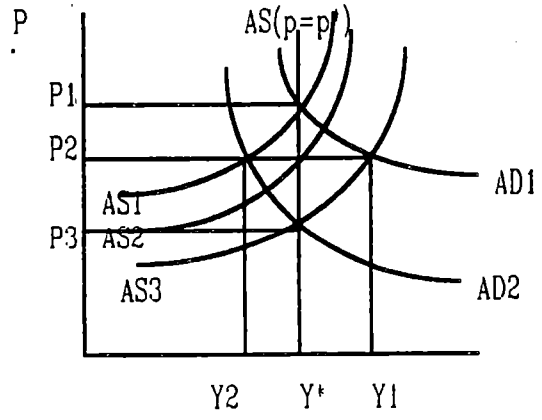
The IS curve shifts to IS1. At (r0,Y1) there is disequilibrium in the money market and interest rates rise, dampening output to Y2. But at (p0,Y2) there is excess demand for goods, so prices are bid up. As a result, the LM curve shifts back as real money balances contract. At the final equilibrium, prices are P1, output is Y3, interest rates rise to r1 and there is an increase in employed labour at N3.



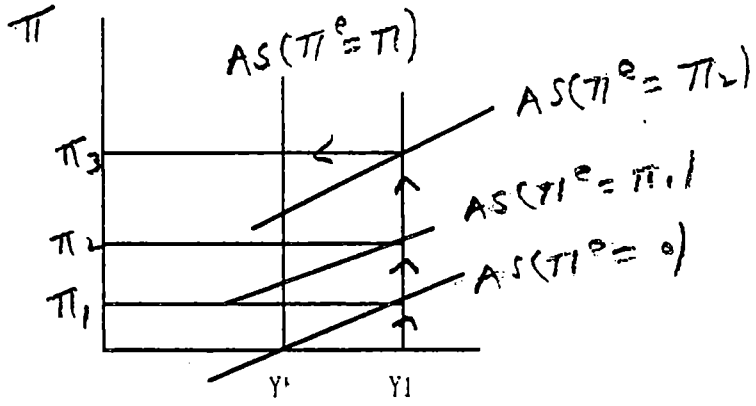
5. Figure 5: IS-LM when AS is vertical. With Y* determined by the production side of the economy, the IS curve determines r*. Given fixed money supply M, the LM curve determines the price level P.



6. Figure 6: Aggregate Demand and Aggregate Supply. Here the vertical AS curve corresponds to those situations where expected price is the same as the true price. The positively sloped AS curves-AS1, AS2, AS3- represent the situations where expected price is given by respectively P1, P2, P3. AS1, for example, crosses the vertical AS curve at P = P1. If demand shifts from AD1 to AD2, the new initial equilibrium is at (P2, Y2). Here it is assumed that P* remains equal to P1. But as price expectations adjust, AS will shift until equilibrium is achieved on AS3 at (P3, Y*).



7. Figure 7: Phillips Curve. This curve is the analogue of Figure 6 with inflation plotted against output. If output is to be maintained at Y1, inflation must continue to increase so that it is greater than expected inflation. If not, and expected inflation equals actual inflation, output will return to Y* but a high rate of inflation.



Lecture 3

Rational Expectations: Optimization With Regard To Available Information

1. Economic agents were depicted in the last two lectures as entirely passive entities who would respond to changes in their environment in a manner completely predicted by the behaviouristic rules formalized by the basic equations of the system. As Newtonian particles, they can be buffeted about by policy, presumably for their own good, and are entirely unaware of the rules describing their economic universe. Surely, by contrast, rational agents extract as best as possible whatever information is contained in the different economic messages they are continuously receiving and modify their behaviour in response. As we have seen in the analysis of the expectations-augmented Phillips curve, agents will hardly guess that inflation will be 3% when it has been consistently over 8% for many quarters. We do have some insight into tomorrow's affairs, because we know today's and have some idea as to how today becomes tomorrow. Of course, there must always remain uncertainty; the point is that we try to do the best possible in the face of the uncertainty.

This Lecture will begin to analyse the issue of the formation of expectations regarding the evolution of macroeconomic variables. The analysis is normative in so far as it is assumed that agents will form expectations or forecasts in an optimum manner; all relevant information is to be exploited in rational expectations formation. To make these ideas clearer and as precise as possible, it is necessary to import ideas from probability theory to describe a world in which there is uncertainty. We shall then describe optimum behaviour in the context where agents attempt to forecast the future course of events.

It should be emphasized at the outset that the Appendix to this Lecture, where the basic ideas of probability are presented is important to the development of the ideas in the main part of the Lecture.

2. To sharpen ideas about best possible forecasting, it is useful to have some simple stochastic or random processes on hand. A basic building block is the sequence which contains minimal information, called "white noise" by engineers. Let $e_1, ..., e_T$ be a sequence of random variables which has the properties:

$$E(e_t) = 0 , \quad \text{for all t.} \tag{1}$$

(the mean of each element in the process is zero);

$$E(e_t^2) = \sigma^2, \quad \text{for all t.} \tag{2}$$

(the variances of the elements are constant);

$$E(e_t e_s) = 0 , \quad \text{for t} \neq \text{s.} \tag{3}$$

(the elements of the sequence are not correlated).

A sequence with these properties represents so much noise or unrelated activity around the constant middle 0; see Figure 1a.. The e_t can be used to build up somewhat more complicated series. One such group is referred to as moving average or MA processes.

MA(1):

$$X_t = b_0 + e_t + b_1 e_{t-1} ; \tag{4}$$

MA(2):

$$X_t = b_0 + e_t + b_1 e_{t-1} + b_2 e_{t-2} ; \tag{5}$$

MA(3):

$$X_t = b_0 + e_t + b_1 e_{t-1} + b_2 e_{t-2} + .. + b_q e_{t-q} . \tag{6}$$

The properties of these series can be inferred directly from their underlying structures and the properties of the building blocks e_t. For example, we compute the mean and variance of an MA(2) process or (5) as follows using the usual rules for manipulating expectations as described in the Appendix..

$$E(X_t) = E(b_0) + E(e_t) + E(b_1 e_{t-1}) + E(b_2 e_{t-2}) = b_0 ;$$

$$Var\,(X_t) = Var\,(b_0) + Var\,(e_t) + Var\,(b_1 e_{t-1}) + Var\,(b_2 e_{t-2}) = (1 + b_1^2 + b_2^2)\sigma^2 .$$

For (5), the series is constructed from a current disturbance or shock e_t and weighted values of past shocks e_{t-1} and e_{t-2} ; the process has a memory of two periods only. An MA(q) process has a memory of q periods. By contrast, autoregressive or AR processes have infinite memory.

AR(1):

$$X_t = a_0 + a_1 X_{t-1} + e_t \; ; \tag{7}$$

AR(2):

$$X_t = a_0 + a_1 X_{t-1} + a_2 X_{t-2} + e_t \; ; \tag{8}$$

AR(p):

$$X_t = a_0 + a_1 X_{t-1} + a_2 X_{t-2} + , ..., + a_p X_{t-p} + e_t \; . \tag{9}$$

(On the assumption that the AR(1) given by (7) has constant mean μ_X and variance σ_X^2 , we see:

$$E(X_t) = E(a_0) + a_1 E(X_{t-1}) + E(e_t) \Rightarrow \mu_X = \frac{a_0}{1 - a_1} \qquad \text{and}$$

$$Var(X_t) = Var(a_0) + Var(a_1 X_{t-1}) + Var(e_t) \Rightarrow \sigma_X^2 = \frac{\sigma^2}{1 - a_1^2} \; .$$

Here, the mean and variance exist only if the autoregressive parameter a_1 is not equal to 1; if a_1 is 1, the series is called a random walk which is of particular interest in contemporary macroeconomic modelling as we will see in the next section of this Lecture.

These series can be presented quite succinctly using the lag operator L, defined as $LX_t = X_{t-1}$. It is clear that $L^2 X_t = X_{t-2}$. MA and AR processes can be written as polynomials in L in a straightforward manner.

MA(q):

$$\begin{aligned} X_t &= b_0 + (1 + b_1 L + b_2 L^2 + ... + b_q L^q) e_t \; , \\ &= b_0 + \varphi(L) e_t \quad ; \end{aligned}$$

where $\varphi(L) = (1 + b_1 L + b_2 L^2 + ... + b_q L^q)$.

AR(p):

$$(1 - a_1 L - a_2 L^2 - ... - a_p L^p) X_t = a_0 + e_t$$

or

$$\psi(L) = a_0 + e_t \; ;$$

where $\psi(L) = (1 - a_1 L - a_2 L^2 - ... - a_p L^p)$.

Polynomials in L can be manipulated much like the more usual polynomials in x. For example, consider the MA(1) process, $X_t = b_0 + e_t + b_1 e_{t-1}$ i.e. $X_t = (1 - b_1 L)e_t$. It follows that $(1 - b_1 L)^{-1} X_t = e_t$. But

$$(1 - b_1 L)^{-1} = 1 + b_1 L + b_1^2 L^2 + b_1^3 L^3 + ... + b_1^n L^n + ... \ ,$$

and so an MA(1) process has an infinite order AR representation.

The two notions introduced in this section can be readily combined. An ARMA(p,q) process (with 0 mean) is given by the following expression:

$$\psi(L) X_t = \varphi(L) e_t \ ,$$

where $\psi(L)$ and $\varphi(L)$ are polynomials in L of order p and q respectively.

3. The models introduced in the previous section are undoubtedly simple, having very little structure. But it is important to realize that they are able to express complicated patterns . For example, consider Figure 1 which shows simulations of an MA(1) process with $b_0 = 0$ and $b_1 = 0.8$ Figure 1a ,and an AR(1) process with $a_0 = 0$ and $a_1 = 1$ Figure 1b ;in both cases the underlying disturbances are independent normal with mean 0 and variance equal to 25. Such simulations are straightforward to implement on a computer, since most statistical software packages have a random number generator giving independent draws from a standard normal distribution and these can be scaled to simulate normals with arbitrary means and variances. The fact that the autoregressive process has a longer memory is reflected in these pictures by the tendency the series has to move away from 0; of course, the mean of the series is 0, so it will return to 0, but the series does not bounce around as much as the moving average example. More complicated looking series can be portrayed by increasing the order of the process; Figure 2 gives examples of MA(2) and AR(2) processes.

In applied work, the researcher begins with a series such as in Figure 3 which gives the quarterly interest rate spread (ie, difference between long and short rates) in the United Kingdom between 1952 and 1988. The goal is to find the simplest ARMA model that describes this data. A common methodology is due to Box and Jenkins which describes a three-step procedure for (a) identifying the order of the model; (b) estimating the model;

and (c) deciding whether the estimation is adequate or whether it can be improved upon. For the example in Figure 3, the simplest or most parsimonious model is an AR(1) with $a_0 = 0.176$ and $a_1 = 0.856$. The added coefficients in an AR(2) or an ARMA(1, 1) two models which the next level of complexity are not significant. This model then could be used for forecasting purposes or as a benchmark against which other forecasts could be assessed.

It must be added that ARMA models by themselves are not able to portray most economic time series whose most important feature is that they exhibit growth over time. For example, consider Figure 4 which a UK stock index over twenty-five years. The series shows an underlying tendency to grow with considerable fluctuation around this growth. How best to model such a process? The traditional approach has been to model the growth component as a simple function of time perhaps to be explained subsequently in a theory of economic growth and to model the fluctuations around this trend growth as an ARMA model. Figure 5 shows a white noise process added to a linear trend M and to a quadratic trend M .Of course, more pronounced fluctuations could have been charaterized by making e_t a more complicated ARMA process. Another approach is suggested by Figure 6 which simulates a simple random walk process $x_t = 2 + x_{t-1} + e_t$, with e_t distributed as NID(0, 9). Here there is no underlying growth component beyond the fixed intercept; none the less, the series exhibits the growth behaviour characteristic of macroeconomic time series. Whether to model such series in this way as incorporating a random walk or to to search for an underlying time trend is a matter of intensive current research which has lead to a variety of other issues and the development of a host of new statistical modeling techniques.

4. Consider a simple process given by: $X_t = X + e_t$, where X is not known. Suppose at time $t-1$ the forecaster wishing to predict X_t given what is known at time $t-1$, offers the forecast $X_t^e = X_{t-1}$. On the face of it, this forecast is quite sensible since agents are familiar with this level of the X-variable. None the less, how should the forecast be evaluated?

First, it is reasonable to define the error associated with this forecast:

$$ERR_t = X_t - X_t^* = X_t - X_{t-1} .$$

It should be emphasized that the error is only known in the subsequent time period t. The error itself is a random variable, and we can ask what is its mean and variance. Of course, the error associated with good forecasting should on average be zero, so we compute for the example:

$$ERR_t = X_t - X_{t-1} = (X + e_t) - (X + e_{t-1}) = e_t - e_{t-1} ,$$

which has mean given by $E(ERR_t) = 0$. Notice that the expectation is computed by inserting the true values of X_t and X_{t-1} in the expression for the error. Here we say that the forecast is *unbiased* which is decidedly a good property. Of course, the forecast should also be tight or close to the true value; so the variance of the error should be as small as possible. The variance can quickly be computed by using trhe rule that it is the sum of expected square value minus the square of the expected value:

$$Var\,(ERR_t) = E(ERR_t^2) - E(ERR_t)^2 = E(e_t - e_{t-1})^2 = 2\sigma^2 .$$

Is it possible to do better in the sense of making a prediction which is unbiased as well but has lower variance? Suppose we consider $X_t^e = \frac{X_{t-1}+X_{t-2}}{2}$, the average of the last two period's realizations of the X-variable. Here the error associated with this forecast is computed:

$$\begin{aligned} ERR_t &= X_t - \frac{(X_{t-1} + X_{t-2})}{2} \\ &= (X + e_t) - \frac{(X + e_{t-1} + X + e_{t-2})}{2} \\ &= e_t - \frac{(e_{t-1} + e_{t-2})}{2} \end{aligned}$$

It follows that $E(ERR_t) = 0$, so that the forecast is unbiased and also

$$Var(ERR_t) = \sigma^2 + \frac{2\sigma^2}{4} = \frac{3\sigma^2}{2} < 2\sigma^2 .$$

It should be clear that if the average of the last three period's realizations of the X-variable were taken as the forecast, the procedure would again be unbiased and that the variance would be:

$$Var(ERR_t) = \sigma^2 + \frac{3\sigma^2}{9} = \frac{4\sigma^2}{3} .$$

A natural procedure then would be to take as best forecast the average of as many past values of the X-variable as are available. The multi-period average is unbiased and the associated forecast error has variance approaching

σ^2 . Moreover, this approach is best possible. Any forecast procedure must lead to error with variance at least as large as σ^2 , since the error term e_t is irreducible .ie. can not be forecast .

5. Suppose the process generating X_t is AR(1): $X_t = X_{t-1} + e_t$. The best guess of X_t given the information at time $t-1$ is evidently X_{t-1}. This procedure is unbiased and has error variance given by σ^2 . Suppose a two-period average had been proposed as the forecast. To see why this approach is sub-optimal, compute

$$\begin{aligned} ERR_t &= (X_{t-1} + e_t) - \left(\frac{X_{t-1} + X_{t-2}}{2}\right) \\ &= X_t - \left(\frac{X_{t-1} + X_{t-2}}{2}\right) \\ &= e_t + \frac{et - 1}{2} \end{aligned}$$

The variance of the error term is evidently greater than σ^2 .

Moral of the tale of two forecasts: In attempting to predict a value of the process X_t based on its past values, the forecaster should attempt first to identify the nature of the process defining the X-variable and to use this information rather than applying a general prediction formula.

6. Signal Extraction. Suppose agents know that the log of nominal wages follows the process

$$w = z + u \,,$$

where z and u are random variables. The agents also know that $\mathrm{E}(z) = E(u) = 0$, and that there is no correlation between u and z, i.e. $E(uz) = 0$. The variances are assumed known to the agent with $E\left(u^2\right) = \sigma_u^2$ and $E\left(z^2\right) = \sigma_z^2$. Prices (in logs) are described by a similar process given by

$$p = z + v \,,$$

where v is a random variable with $E(v) = E(zv) = E(uv) = 0$. Wages and prices share z which represents economy-wide changes in wages and prices; the variable u represents a shock to wages, while v is an independent shock to prices. The agent sees only w and p, but can not de-compose these signals into their constituents z, u and p , with the result that he can not

be certain about the true real wage rate. In this context, how is the agent to evaluate a particular signal w describing the nominal wage rate?

Aside. Notice that this rather simple model gives some substance to the labour supply decision described in the previous Lecture. Here, as there, labour simply does not know the real wage rate. A decision must be taken based on the fluctuating nominal wgae rate in the context of fluctuating prices. To the extent that labour is systematically wrong, aggregate supply will deviate from the natural level associated with the production function at which the labour market clears and there is no price illusion.

To make the problem in this context at all tractable, let us suppose that the agent wishes to forecast the real wage level $w - p$ as a linear function of the nominal wage signal w:

$$w - p = a_0 + a_1 w \, . \tag{10}$$

As in the previous section, it seems reasonable to adopt as a choice criterion for the parameters a_0 and a_1 the condition that the variance of the forecast error is minimized. More formally,

$$\min \; E \;\; (w - p) - (a_0 + a_1 w) \;\; ^2$$

There are two first-order conditions corresponding to the two parameters a_0 and a_1 ; the derivative with respect to a_0 is:

$$-2\,E\;(w - p) - (a_0 + a_1 w) \;= 0 \, . \tag{11}$$

Substituting for $w - p$ and w, we have

$$E\;(u - v) - (a_0 - a_1(z + u) \;= 0 \, ;$$

from which it follows that $a_0 = 0$, since the expected value of each of l the random variables is 0. Now, taking the derivative with respect to a_1, we find

$$-2\,E \;\; (w - p) - (a_0 + a_1 w) \;\; w = 0 \Rightarrow E \;\; (u - v) - (a_0 - a_1(z + u))(z + u) \;= 0 \, . \tag{12}$$

Only expectations of cross terms are non-zero; from which it follows:

$$a_1 = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_z^2} \quad . \tag{13}$$

The expression for the best forecast for the real wage rate given the nominal wage signal w is $a_1 w$ as in (10) with a_1 given by 13. To motivate the result which involves a ratio of variances, suppose that the variance of the economy-wide disturbance is much greater than the variance of the local disturbance u; according to (13), the best response to the signal w is close to 0: the agent should attribute interpret increases in w as part of a general shift in prices and not as an increase in his real wage. By contrast, if the variance of u is much greater than σ_z^2, the agent is best advised to interpret w as an increase in real wages, i.e. $a_1 = 1$.

7. The argument of the previous section can be re-cast in the following more general framework. Consider two random variables y and x with known first moments i.e. $E(x)$ and $E(y)$ and known second moments i.e. $E(x^2)$, $E(y^2)$ and $E(xy)$. The linear projection $P\ y \mid 1, x$ is given by $a_0 + a_1 x$, where the parameters are chosen to minimize the expression:

$$E\ \ y - P\ y \mid 1, x\ ^2\ .$$

The corresponding first-order conditions are called the *normal equations* associated with the projection:

$$E\ \ y - (a_0 + a_1 x)\ = 0\ , \tag{14}$$

$$E\ \ (y - (a_0 + a_1 x))x\ = 0\ . \tag{15}$$

From (14) it follows that $a_0 = E(y) - a_1 E(x)$; substituting this expression for a_0 into (15), we obtain:

$$E(xy) - E(x)E(y) + a_1 E(x)^2 - a_1 E(x^2) \Rightarrow a_1 = \frac{Cov(x,y)}{Var(x)}\ ,$$

by the rules developed in the Appendix. Hence we can write:

$$P\ \ y \mid 1, x\ = \frac{Cov(x,y)}{Var(x)} x$$

The values for the parameters found in this way are sometimes called the population regression coefficients. These should not be confused with the standard regression coefficients defined by the sample means, variances and covariances.

This presentation can readily be extended to define the expression P $y \mid 1, x_1, ..., x_n$ as a linear projection of y on the variables $x_1, ..., x_n$ given the first and second moments of the random variables in question.

8. In early empirical work on the Keynesian consumption, researchers considered regressions of the form:

$$C = a + bY \tag{16}$$

and found that $a > 0$ with the implication so that the average propensity to consume (APC) exceeds the marginal propensity to consume; see Figure 7. MPC in the diagram is given by b, whereas the APC at Y_0 is given by the slope of the ray OX from the origin. However, in long-run historical data, APC is found to equal MPC. How to reconcile these results?

One way of looking at Milton Friedman's permanent income hypothesis is as a signal extraction problem. His model can be characterized as follows.

$$C_i = \beta Y_{pi} + e_i \,, \tag{17}$$

$$Y_i = Y_{pi} + Y_{Ti} \quad . \tag{18}$$

Here Y_i and C_i are observable income and consumption of the ith household, while Y_{pi} and Y_{Ti} represent the permanent and transitory components of income. For the researcher the variable Y_{pi} is not directly observable. Friedman assumed that $E(e_i) = E(Y_{Ti}) = 0$, while there is no correlation among the components of income and the disturbance to consumption:

$$E(e_i Y_{pi}) = E(e_i Y_{Ti}) = E(Y_{pi} Y_{Ti}) = 0 \,.$$

Notice that consumption is directly proportional to permanent income in Friedman's model with the consequence that APC = MPC, a result which is confirmed by the historical data .

Suppose in the context of Friedman's model that observable consumption C_i is projected on 1 and observable income Y_i. According to the results of the previous section, we have

$$P \quad C_i \mid 1, Y_i \; = h_0 + h_1 Y_i \,,$$

where $h_0 = E(C_i) - h_1 E(Y_i)$ and $h_1 = \frac{Cov(C_i, Y_i)}{Var(Y_i)}$. From (18)

$$Var(Y_i) = Var(Y_{pi}) + Var(Y_{Ti}) \,,$$

which follows, since there is no correlation between permanent and transitory income. Using (17) and (18),

$$\begin{aligned} Cov(C_i, Y_i) &= E\ \ (C_i - E(C))(Y_i - E(Y)) \\ &= E\ \ (\beta Y_{pi} - E(Y_p))(Y_{pi} - E(Y_p)) + Y_{Ti} \\ &= \beta Var(Y_{pi}) \end{aligned}$$

It follows that

$$h_1 = \frac{\beta\,(Var(Y_{pi}) + Var(Y_{Ti}))}{Var(Y_{pi})} < \beta\ ,$$

as long as $Var(Y_{Ti}) \neq 0$ so that the usual regression estimate will underestimate the true population co-efficient. Moreover,

$$h_0 = E(C_i) - h_1 E(Y_i) = \beta E(Y_{pi}) - h_1 E(Y_{pi}) = (\beta - h_1) E(Y_{pi}) > 0\ .$$

The regression intercept is positive within the context of the Friedman model even though consumption is directly proportion to permanent income.

Appendix: The Rudiments of Probability Analysis

A1. The probability density function $f(x)$ is characterized by the two properties:

a) $f(x) \geq 0$, for all x;
b) $\int_{-\infty}^{+\infty} f(x)dx = 1$.

Note: An integral with infinite limits is calculated by first computing $\int_b^a f(x)dx$ and then letting $a \to \infty$ and $b \to \infty$.

A random variable X can be associated with $f(x)$. Let A be a subset of the real numbers .i.e. $A \subseteq R$; A is called an event and the probability that the random variable X occurs as an A is given by the expression:

$$\Pr \quad X \in A = \int_A f(x)dx .$$

In this way, probability is associated with the area underneath the density function.

Examples:

1. Uniform distribution. Suppose that $f(x) = 1/4$ when x is between 0 and 4, and is 0 elsewhere. First, $f(x)$ is indeed a probability density function as shown by the following calculation.

$$\int_{-\infty}^{+\infty} f(x)dx = \int_0^4 \frac{1}{4}dx = \left|\frac{x}{4}\right|_0^4 = 1 - 0 = 1 .$$

2. Let Z represent the standard normal random variable ; $Z \sim N(0,1)$, where the mean μ is 0 and the variance σ^2 is 1. The density function is given by:

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-x^2}{2}\right\}$$

To find the probability that a draw from the standard normal lies between 1/4 and 3/4, i.e. Pr $1/4 \leq Z \leq 3/4$,we compute the integral of $f(x)$ given above from 1/4 to 3/4 or, more simply, look up the normal table at the back of a statistics text.

A2. In many circumstances, it is necessary to analyse the joint behaviour of two random variables X and Y. What, for example, is the probability that X is between 0 and 1 while Y is between 2 and 3? The definition of the density function must be modified. The two basic properties are:

$$\begin{array}{lll} a) & f(x.y) \geq 0\ , & \text{for all x,y} \\ b) & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x;y)dydx = 1 & \end{array}$$

View (b) as

$$\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x,y)dy \right) dx = 1\ .$$

and compute the inner integral treating x not as a variable but as a constant. According to mathematical terminology, we first integrate out the y- variable then the x-variable. The following example, which will be used extensively to elaborate the various concepts introduced in what follows, illustrates the basic sequential idea in computing a multiple integral.

Example 3. Define $f(x,y) = 2$ when$0 < x < y < 1$, with $f(x,y) =$ 0elsewhere. To show that $f(x,y)$ is indeed a density, we compute $\int_0^1 \left(\int_x^1 2dy \right) dx$. First, the inner integral is simply: $|2y|_1^x = 2(1-x)$. So we have

$$\int_0^1 \left(\int_x^1 2dy \right) dx = \int_0^1 \quad 2(1-x) \quad dx = |2x|_0^1 - \left|x^2\right|_0^1 = 2 - 1 = 1\ .$$

A3. Define the expectations operator as follows:

$$E\ g(x,y)\ = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} g(x,y)\, f(x,y)dy \right) dx$$

Some standard examples of this notion include the following.

1. mean of X :

$$\mu_X = E(X) = \int_{-\infty}^{+\infty} x\, f(x)dx\ ;$$

2. Variance of X:

$$\sigma_X^2 = Var(X) = E(X - \mu_X)^2 = \int_{-\infty}^{+\infty} (x - \mu_X)^2\, f(x)dx$$

3. Covariance between X and Y:

$$\sigma_{XY} = Cov(X,Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(x-\mu_X)(y-\mu_Y)f(x,y)dydx$$

The commonly used properties of these notions follow directly from the basic definitions and the additivity property of the integral. Letting X, Y stand for random variables and a, b for constants, we have:

4. $E(aX+bY) = aE(X)+bE(Y)$
5. $Var(aX+b) = a^2 Var(X)$
6. $Var(aX+bY) = a^2 Var(X)+b^2 Var(Y)+2abCov(X,Y)$

Using (4). it is straightforward to show the following useful practical way of computing the variance of X:

$$Var(X) = E(X-\mu_X)^2 = E\ X^2 - 2\mu_X X + (\mu_X)^2 = E(X^2) - (\mu_X)^2\ ,$$

since

$$\int_{-\infty}^{+\infty}\mu_X\, x\, f(x)dx = (\mu_X)^2$$

Similarly,

$$Cov(X,Y) = E(XY) - E(X)E(Y)\ .$$

Example 3 (cont'd)

The mean and variance of the uniform distribution described in Example 3 are computed as follows.

$$\mu_X = \int_0^4 x\left(\frac{1}{4}\right)dx = \left|\frac{x^2}{8}\right|_0^4 = 2 \qquad \text{and}$$

$$E(X^2) = \int_0^4 x^2\left(\frac{1}{4}\right)dx = \left|\frac{x^3}{12}\right|_0^4 = \frac{16}{3}\ ,$$

so that

$$\sigma^2 = \frac{16}{3} - 4 = \frac{4}{3}\ .$$

A4. Given the bivariate density function $f(x,y)$ for the random variables X and Y, several important univariate density functions can be extracted.

The first are the marginal distributions corresponding to X alone and to Y alone. Define:

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy \qquad \text{and} \qquad f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)dx \ ;$$

These are called the marginal distributions associated with respectively X and Y. In one case, the variable y is integrated out and in the other the variable x. These correspond to the question: what is the density of X at x irrespective of the value of Y (alternatively, what is the density of Y at y irrespective of the value of X). With the univariate notion of marginal distribution in hand, the idea of the conditional distribution can be introduced as follows:

$$f_{X \mid Y=y}(x) = \frac{f(x,y)}{f_Y(y)} \qquad \textit{and} \qquad f_{Y \mid X=x}(y) = \frac{f(x,y)}{f_X(y)} \ .$$

The first definition corresponds to the density of X at x when it is given that $Y = y$, while the second is the density of Y at y given that $X = x$. Sometimes these densities are written more simply as $f(x \mid Y = y)$ and $f(y \mid X = x)$, or even $f(x \mid y)$ and $f(y \mid x)$. It is important to realize that these densities are indeed probability densities and, as such, have means and variances. We return to an extended consideration of Example 3 to illustrate these notions.

A5. Recall that in Example 3 $f(x,y) = 2$ as long as $0 < x < y < 1$, and is 0 elsewhere. We first compute the univariate marginal distributions corresponding to this bivariate density.

$$f_X(x) = \int_1^x 2dy = |2y|_1^x = 2(1-x) \ ;$$

$$f_Y(y) = \int_0^y 2dx = |2x|_0^y = 2y \ .$$

Notice that

$$\Pr\left[0 < x < \frac{1}{2}\right] = \int_0^{\frac{1}{2}} 2(1-x)dx = \left|2x - x^2\right|_0^{\frac{1}{2}} = 1 - \frac{1}{4} = \frac{3}{4} \ .$$

or that

$$\mu_Y = \int_0^1 2y^2 dy = \left|\frac{2y^3}{3}\right|_0^1 = \frac{2}{3} \ .$$

Similarly, it can be shown that $\mu_X = 1/3$ and that $E(X^2) = 1/6$, from which it follows that $Var(X) = E(X^2) + E(X)^2 = 1/18$. To compute $Var(Y)$, first observe that $E(Y^2) = 1/2$, so that $Var(Y) = E(Y^2) - (\mu_Y)^2 = 1/18$.

Now, using the marginal distributions, the conditional distributions can readily be computed. For example, for $x < y$,

$$f_{X \mid Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{2}{2y} = \frac{1}{y}\,; \quad f_{Y \mid X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{1}{1-x}\,.$$

Recall that $\Pr\, 0 < X < 1/2 = 3/4$. Suppose we know that $Y = 3/4$; what is $\Pr\, 0 < X < 1/2 \mid Y = 3/4$? We compute directly from the conditional distribution: .

$$\int_0^{\frac{1}{2}} \frac{1}{3/4} dx = \left| \frac{4x}{3} \right|_0^{\frac{1}{2}} = \frac{2}{3}\,.$$

We conclude the analysis of this bivariate density by calculating $Cov(X, Y)$. First observe,

$$\begin{aligned} E(X,Y) &= \int_0^1 \left(\int_x^1 2xy dy \right) dx \\ &= \int_0^1 (x - x^3) dx \qquad \left(\text{since } \int_x^1 2xy dy = \left| xy^2 \right|_x^1 = x - x^3 \right) \\ &= \left| \frac{x^2}{2} - \frac{x^4}{4} \right|_0^1 \\ &= \frac{1}{2} - \frac{1}{4} \\ &= \frac{1}{4} \end{aligned}$$

Now, since $Cov(X,Y) = E(XY) - E(X)E(Y)$, it follows that $Cov(X,Y) = 1/4 - 2/9 = 1/36$.

One last technical idea. The correlation coefficient ρ_{XY} is defined as:

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}\,.$$

. This number is always between -1 and 1, with 0 representing no correlation between X and Y. In our example, ρ_{XY} is given by $\frac{1/36}{\sqrt{(1/18)}\sqrt{(1/18)}} = 0.5$.

Exercises

1. Suppose X_t follows an MA(3) process with mean b_0 . Compute the following expectations:

$$E(X_t^2), Var(X_t), E(X_t, X_{t-1}), E(X_t, X_{t-2}), E(X_t, X_{t-3}), E(X_t, X_{t-4}) .$$

2. Suppose X_t follows an AR(2) process with mean 0. Compute the following expectations:

$$E(X_t^2), Var(X_t), E(X_t, X_{t-1}), E(X_t, X_{t-2}) .$$

3. Suppose Y_t follows an MA(1) process with mean 0. Compute $P\ Y \mid 1, X$, where X_t is the lag of Y_t ; ie $X_t = Y_{t-1}$.

4. Repeat problem 3 when Y_t follows an (a) AR(1) process with mean 0; (b) AR(2) process with mean 0.

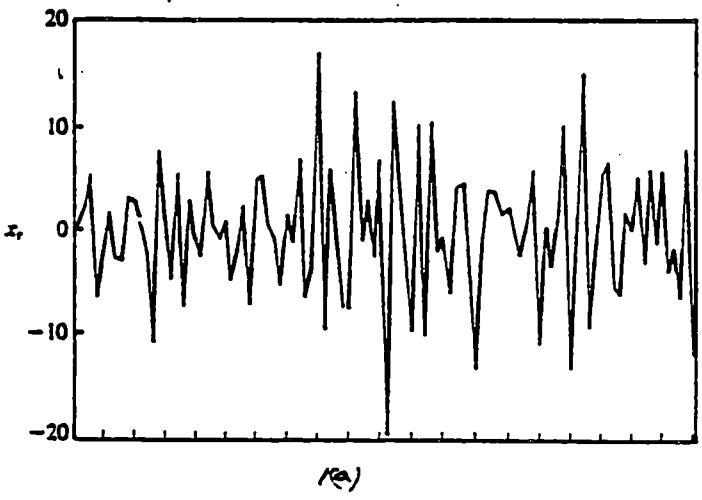
5. An example from Hogg and Cragg.

6. Term structure at short end.

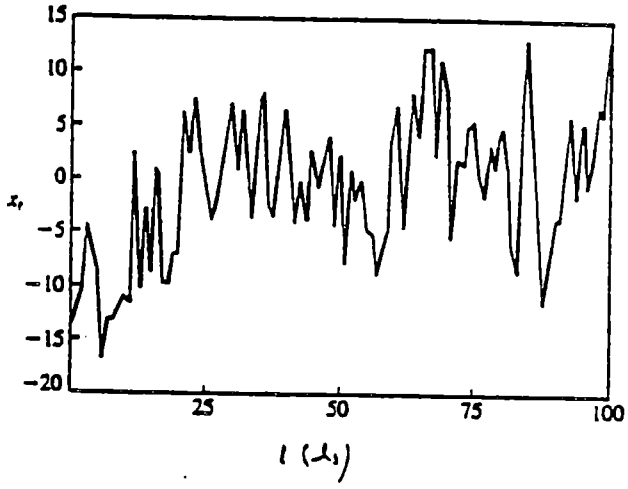
LECTURE 3

Graphs

1. Figure 1a: simulation of an MA(1) process with mean zero and $b_1 = 0.8$. Figure 1b: simulation of an AR(1) process with mean 0 and $a_1 = 0.5$. The underlying disturbances are NID(0, 25) [ie, independently and identically normal with mean 0 and variiance 25].

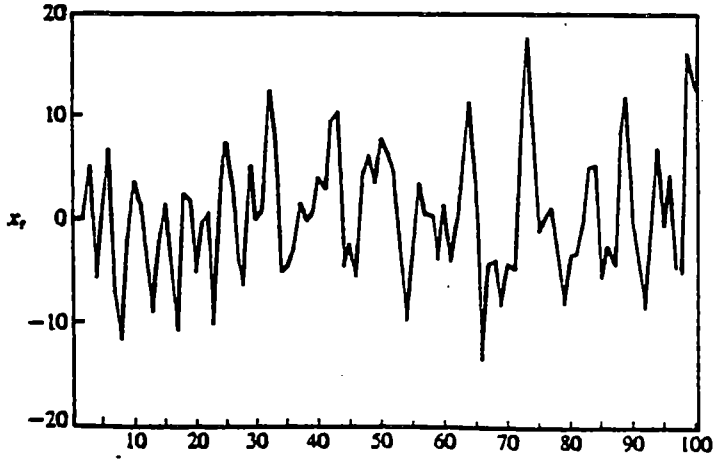


1(a)

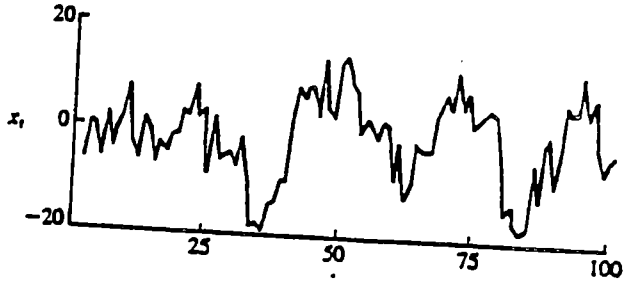


1 (b)

2. Figure 2a: simulation of an MA(2) process with mean zero and $b_1 = -0.5$, $b_2 = 0.3$. Figure 1b: simulation of an AR(2) process with mean 0 and $a_1 = 0.5$, $a_2 = 0.3$. The underlying disturbances are NID(0,25).

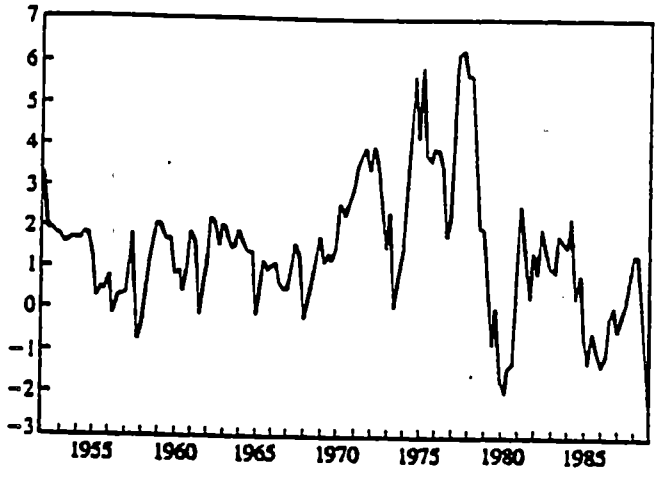


2 (a)

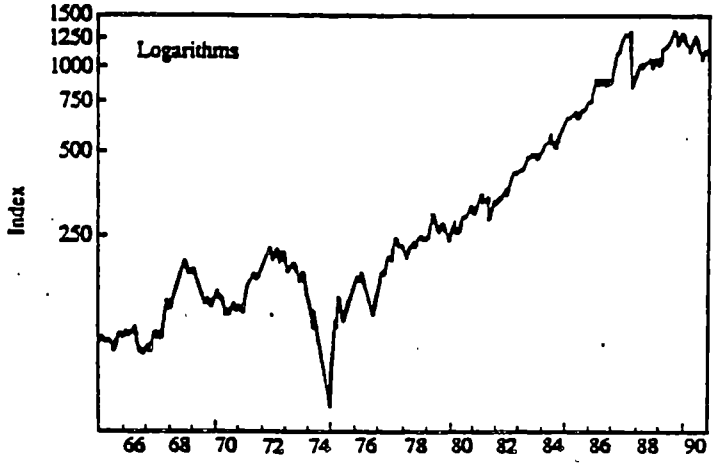


2 (b)

3. Figure 3. Quarterly interest rate spread [UK , 1952 - 1988].

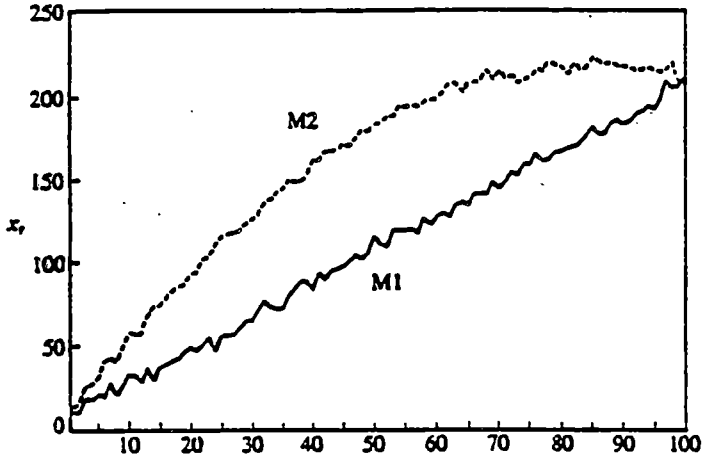


4. Figure 4: FTA Share Index [UK, monthly 1965 - 90].

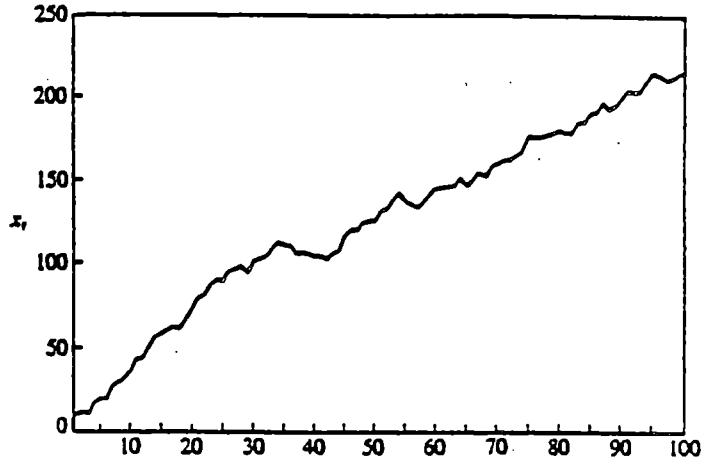


5. Figure 5: Linear Trend given by M1, generated by $x_t = 10 + 2t + e_t$; and Quadratic Trend given by M2, generated by $x_t = 10 + 5t - 0.03t^2 + e_t$. The e_t are NID(0,9).

Figure 6: A simple random walk model : $x_t = 2 + x_{t-1} + e_t$. The e_t are NID(0,9).

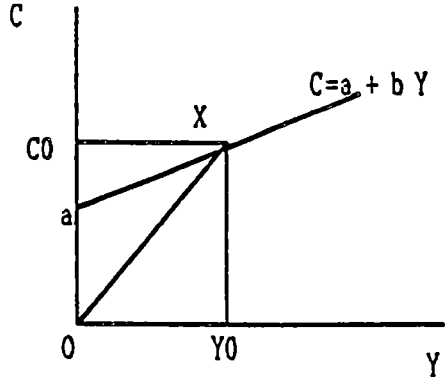


(5)



(6)

6. Figure 7. Suppose consumption C is linear in Y with slope b and non-zero intercept a. At Y0, the marginal propensity to consume (MPC) is given by b, whereas the average propensity to consume (APC) is C0/Y0 (or the slope of OX).



Lecture 4

Rational Expectations Approach To Model Solving
Policy Ineffectiveness

1. Introduction

2. Consider the following stripped-down AS-AD monetary model, where the first equation describes the demand for money, the second presents the supply as centered around m^* and the third describes a Phillips curve relationship with y^* denoting equilibrium output. All variables are in logarithmic form.

$$m_t = p_t + y_t \tag{1}$$

$$m_t = m^* + e_t \tag{2}$$

$$(\, p_t - E_{t-1}p_t) = \alpha(\, y_t - y^*) \tag{3}$$

Here the endogenous variables are m_t, y_t and p_t, while the exogenous variables are m^* and the price expectations term E_{t-1} . Using (1) and (2), we can readily eliminate m_t.

$$p_t + y_t = m^* + e_t \tag{4}$$

The remaining endogenous variables can be found by successive elimination. First, substitute (4) into (3) to eliminate y_t.

$$p_t = E_{t-1}p_t + \alpha \;\; (m^* - p_t + e_t) - y^* \;, \text{ or}$$

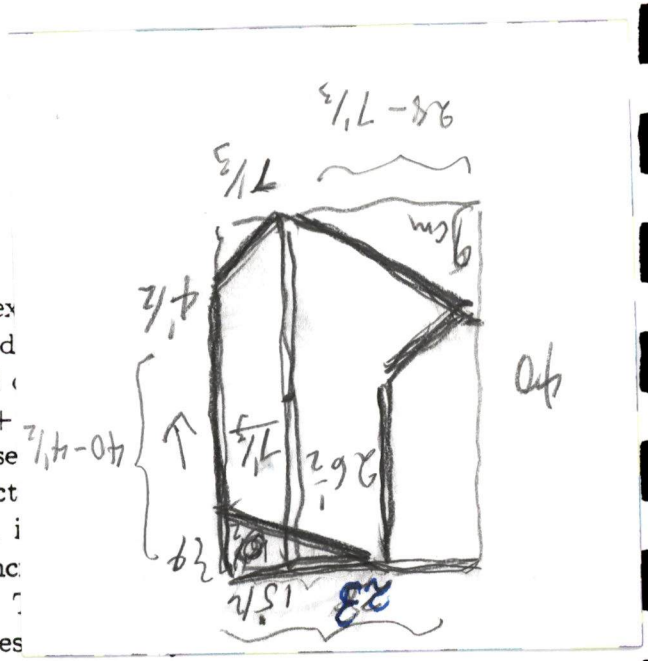
$$(1+\alpha)p_t = E_{t-1}p_t + \alpha(m^* - y^*) + \delta\, e_t \;. \tag{5}$$

Output can be determined in a similar way.

$$y_t = m^* - \;\; E_{t-1}p_t + \alpha(\, y_t - y^*) \;\; + e_t \,,$$

which implies

$$(1+\alpha)y_t = m^* + \alpha y^* - E_{t-1}p_t + e_t \;. \tag{6}$$



The solution to the model with exogenous ex
by (5) and (6). As usual, for this sort of mod
the government can affect the equilibrium level
the money supply m^* ; here $\partial y_t/\partial m^* = 1/(1+$
prices change as well as a result of the increase
seems unreasonable to assume that price expect
if agents know that m^* will increase. In fact, i
assume that agents do know that m^* will be inc
the price mechanism is governed by (1) - (3).
we take expectations of the solution (5) for prices
analysis of the model.

$$(1+\alpha)E_{t-1}p_t = E_{t-1}p_t + \alpha(m^* - y^*) \; . \tag{7}$$

Here we have assumed that (i) the expectations are linear; (ii) $E_{t-1}\, aX = aE_{t-1}X$; (iii) $E_{t-1}a = a$; and (iv) $E_{t-1}e_t = 0$. The first three properties of mathematical expectation were established in the Appendix to the previous lecture, while (iv) is an assumption of the model. From (7),

$$\alpha E_{t-1}p_t = \alpha(m^* - y^*) \Rightarrow E_{t-1}p_t = m^* - y^* \tag{8}$$

The importance of the argument can not be over-emphasized: price expectations have been made endogenous to the model. Equilibrium levels of y_t and p_t can now be determined from the previous solutions (5) and (6). Start with prices:

$$(1+\alpha)p_t = (m^* - y^*) + \alpha(m^* - y^*) + \alpha\, e_t \; ,$$

or

$$p_t = m^* - y^* + \alpha\, e_t \tag{9}$$

Now consider output.

$$(1+\alpha)y_t = m^* - (m^* - y^*) + \alpha\, y_t + e_t \; ,$$

$$y_t = y^* + e_t \tag{10}$$

The results given by (8) , (9) and (10) make good sense. Notice first that the forecast error $p_t - E_{t-1}$ associated with the solution is simply $\alpha\, e_t$; if agents know the money supply rule and how the rule is incorporated in this economy, then they will not make any systematic forecast errors, predicting prices correctly up to the unpredictable component based on et . Accordingly, it should not be surprising that equilibrium output will be

given the long-run level y^*. There is no short-run Phillips curve possible in this model when expectations are formed rationally.

3. The solution to rational expectations models can take different approaches according to the nature of the model. Consider the case where the expectations terms in the model involve only current values of the endogenous variables. The model analysed in the previous section is an example. The following simple example will be solved in two ways.

$$y_t = a + b_{t-1}y_t + G^* + e_t \tag{11}$$

Here we denote the expectation of y_t formed at $t-1$ by $_{t-1}y_t$; G^* is a fixed exogenous variable and et represents a disturbance independent of the past with mean 0. As above, we assume that expectations are formed rationally and interpret the expectations term as $E_{t-1}y_t$. This term can now be viewed endogenously simply by taking expectations of (11) conditional on information available at $t-1$.

$$E_{t-1}y_t = a + bE_{t-1}y_t + G^* + 0 \Rightarrow E_{t-1}y_t = \frac{a+G^*}{1-b} .$$

This model-determined value of $E_{t-1}y_t$ can be substituted back into (11) to determine the general solution to the model.

$$y_t = a + b\left(\frac{a+G^*}{1-b}\right) + e_t \Rightarrow y_t = \frac{a+G^*}{1-b} + e_t .$$

Notice here that the expectational error is simply e_t.

A second solution technique termed " the method of undetermined coefficients" now follows. Here we guess that the true solution has the form

$$y_t = k + \pi_0 e_t + \pi_1 e_{t-1} + \pi_2 e_{t-2} + ..., \tag{12}$$

where k is a constant, the e_{t-j} are the model disturbances and the π_i are the solution coefficients to be determined. The strategy is first to take expectations of (12) to determine the value of $E_{t-1}y_t$.

$$E_{t-1}y_t = k + \pi_1 e_{t-1} + \pi_2 e_{t-2} + ..., \tag{13}$$

since $E_{t-1}e_t = 0$. This expression is now substituted in the original model (11).

$$y_t = a + b(k + \pi_1 e_{t-1} + \pi_2 e_{t-2} + ...) + G^* + e_t \tag{14}$$

We now have two expressions (12) and (14) for y_t which are now equated term by term to determine coefficients π_i. Clearly, $k = (a + G^*)/(1 + b)$. Equating terms in e_t we have $\pi_0 = 1$. Looking at e_{t-1}, it follows that $b\pi_1 = \pi_1$ which implies $\pi_1 = 0$, since $0 < b < 1$. Similarly, $\pi_i = 0$ for $i > 2$. The expression (12) now becomes

$$y_t = \frac{a + G^*}{1 + b} + e_t \ ,$$

as before.

4. Consider now a simple model in which there are forward expectations but no lagged variables. Economic examples are given in subsequent sections.

$$y_t = a + bE_t\, y_{t+1} + G_t + e_t \ . \tag{15}$$

We proceed as before. First we "lead" (15) by one period to obtain

$$y_{t+1} = a + bE_{t+1} y_{t+2} + G_{t+1} + e_{t+1} \ .$$

Now we take expectations of this equation conditional on information available at time t.

$$E_t y_{t+1} = a + bE_t\, y_{t+2} + E_t G_{t+1} \ . \tag{16}$$

Here the following fact is used : $E_t\, E_{t+j}\, y_{t+\,j+1} = E_t\, y_{t+i+1}$. The intuition is that at time t agents are using all available information in forecasting y_{t+i+1}. To be sure, additional information may arise between t and $t + i$ which would lead to a revision of the forecast; but this revision occurs at time $t + i$. At time t, the best forecast of y_{t+i+1} is $E_t\, y_{t+i+1}$ and the best forecast of what the forecast will be at time $t + i$ i.e. $E_t E_{t+\,j}\, y_{t+j+1}$ is still $E_t\, y_{t+i+1}$. We now substitute (16) back into (15) and re-arrange terms:

$$y_t = a + b(a + bE_t\, y_{t+2} + E_t G_{t+1}) + e_t \ , \text{or}$$

$$y_t = a + ab + b^2 E_t\, y_{t+2} + G_t + bE_t\, G_{t+1} + e_t \ . \tag{17}$$

One expectations expression has now been replaced by another. An important feature of (17) is the emergence of expectations of future government spending G_{t+1} . We repeat the previous argument by considering the original model at time $t + 2$.

$$y_{t+2} = a + bE_{t+2} y_{t+3} + G_{t+2} + e_{t+2} \ .$$

We now obtain taking expectations again conditional at time t:

$$E_t y_{t+2} = a + bE_t\, y_{t+3} + E_t G_{t+2}\,. \tag{18}$$

And so (17) becomes

$$y_t = a + ab + b^2(a + bE_t\, y_{t+3} + E_t G_{t+2}\,) + G_t + bE_t\, G_{t+1} + e_t\,, \text{ or}$$

$$y_t = a + ab + ab^2 + b^3 E_t\, y_{t+3} + G_t + bE_t\, G_{t+1} + b^2 E_t G_{t+2}\, + e_t\,.$$

Now the expectations terms concern the endogenous variable three periods ahead and the exogenous variable two periods ahead. The emerging pattern of the final solution should be clear. After N periods

$$y_t = a+ab+...+ab^{N-1}+b^N E_t\, y_{t+N}+G_t+bE_t\, G_{t+1}+...+b^{N-1}E_t G_{t+N\,-\,1}+e_t\,.$$

Now let $N \to \infty$,

$$y_t = \frac{a}{1-b} + G_t + bE_t G_{t+1} + b^2 E_t G_{t+2} + ... + e_t\,,$$

assuming that $b^N E_t\, y_{t+N} \to 0$ as $N \to \infty$. The importance of this solution is that it captures the forward-looking character of the model. Expectations of all future levels of government spending will affect current output.

5. The method of undetermined coefficients is now applied to the previous problem. To begin, we guess that the solution takes the form:

$$y_t = h + ke_t + \pi_0 G_t + \pi_1 E_t G_{t+1} + \pi_2 E_t G_{t+2} + ..., \tag{19}$$

so that

$$E_t\, y_{t+1} = h + \pi_0 E_t G_{t+1} + \pi_1 E_t G_{t+2} + \pi_2 E_t G_{t+3} + ... \tag{20}$$

We now substitute this expression for the expectation of future income into (15):

$$y_t = a + b(h + \pi_0 E_t G_{t+1} + \pi_1 E_t G_{t+2} + \pi_2 E_t G_{t+3} + ...) + G_t + e_t. \tag{21}$$

The coefficients of the corresponding terms in (19) and (20) are equated to yield the solution. First, consider the intercept: $h = a + bh \Rightarrow h = a/(1-b)$. Second, a look at the coefficients of e_t and G_t yields immediately that $k = 1$ and $\pi_0 = 1$. The coefficients of $E_t G_{t+1}$ are $b\pi_0$ and π_1, so that $\pi_1 = b$.

Looking at the coefficients of $E_t G_{t+2}$, we see that $b\pi_1 = \pi_2$, so that $\pi_2 = b^2$. We proceed recursively to find that $\pi_i = b^i$. In this way, the solution found in the previous section is verified.

6. Consider the following suggestive application of these methods. Let the price of a stock at time t be denoted by p_t; the dividend by d_t. Suppose also that the riskless rate of return r is constant over time. We also assume in this example that investors are risk averse .The expected rate of return on the stock from one period to the next must, if investors are risk averse, be equal to the riskless rate due to arbitrage. More formally,

$$\frac{E_t p_{t+1} - p_t}{p_t} + \frac{d_t}{p_t} = r \tag{22}$$

The first term in this expression is the expected capital gain, the second is the dividend rate. Re-arranging terms, we find

$$(1+r)p_t = E_t p_{t+1} + d_t \text{ , or}$$

$$p_t = aE_t p_{t+1} + ad_t \text{ ,} \tag{23}$$

where $a = 1/(1+r)$. Now this equation takes precisely the form of (15). It follows that the current price of the stock under the assumed conditions will be a weighted average of current and expected future dividends. The details are left to an exercise.

7. A more complicated model involves forward expectations of some variables with lagged values of other variables.

8. The rational expectations approach to macroeconomic models that contain expectations is simply a comprehensive model solution technique which attempts to determine values of agent's expecations using features of the model itself. The solution need not be unique as the following example involving a speculative bubble illustrates.

Exercises

1. Consider the following model given by:

$$m_t = p_t + y_t$$

$$m_t = m^* - \gamma(\, y_{t-1} - y^*) + e_t \quad \gamma > 0$$

$$(p_t - {}_{t-1}\, p_t) = \alpha(\, y_t - y^*) \qquad \alpha > 0$$

a) Solve the model assuming that expectations of p_t formed in the previous period i.e. ${}_{t-1}p_t$ are given exogenously.
b)Using these results, find the rational expectations solution. Compare your results with those obtained in section 2.

2.Repeat the previous exercise for the following discrete time version of AS-AD model.

$$m_t = p_t + \alpha(y_t - y^*) + g_t \qquad \dot{\alpha} > 0$$

$$y_t = y^* + \delta(p_t - p^e_{t-1}) + e_t \qquad \delta > 0$$

3. Cagan model.

4. Sargent p 202.

Lecture 5

Lucas Supply Curve
A Policy Effectiveness Result

1. Introduction

2. The Lucas Aggregate Supply curve is derived from the optimizing behaviour of many individual suppliers who are producing the same good but who are isolated in separate and isolated individual markets. Demand is distributed unevenly across markets and induces different prices in each market. Each market is identical except for this feature; accordingly, the markets can be indexed by the demand shock z. There are also economy-wide price movements, so the individual producer must distinguish between the local price change or real demand shock and the nominal price movement across the economy. Accordingly, local supply is governed by

$$y_t(z_t) = \gamma\,(p_t(z) - P \quad p_t \mid I_t(z) \quad), \qquad \gamma > 0 \tag{1}$$

where

$y_t(z_t)$: supply in market indexed by demand shock z_t,

$p_t(z)$: price in market that has shock z , p_t : the economy-wide price,

$I_t(z)$: the information set available in market z at time t.

All values are given in logs; output represents deviations from some underlying natural rate.

According to (1), the local producer determines whether the demand price viewed in the local market is different from the expected economy price given the available information. What is this information set I_t? For the moment we assume that the current local market price and all past information, denoted Ω_{t-1}, is known to the supplier in that market; $I_t = \{p_t(z), \Omega_{t-1}\}$. The goal now is derive an aggregate supply curve for the